

# THE ELLIPTIC MATHAI–QUILLEN FORM AND AN ANALYTIC CONSTRUCTION OF THE COMPLEXIFIED STRING ORIENTATION

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**ABSTRACT.** We construct a cocycle representative of the elliptic Thom class using analytic methods inspired by a 2-dimensional free fermion field theory. This produces the complexified string orientation in elliptic cohomology, and hence determines a pushforward for families of rational string manifolds. We construct a second pushforward motivated by the supersymmetric nonlinear sigma model studied by Witten in relation to the Dirac operator on loop space. We show that these two pushforwards agree. Analogous constructions in 1-dimensional field theories produce the Mathai–Quillen Thom form in complexified K-theory and the  $\hat{A}$ -class for a family of oriented manifolds.

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## 1. INTRODUCTION

The last 35 years have seen rich cross-fertilization between topology, geometry, analysis and quantum field theory. A key aspect is rooted in the connection between supersymmetric mechanics and the Atiyah–Singer index theorem [Wit82, AG83, Get88]. Witten’s work from the late 80s [Wit87, Wit88] points towards a generalization of the index theorem associated with the analysis of 2-dimensional quantum field theory on the one hand and the algebraic topology of elliptic cohomology on the other—specifically, the string orientation of topological modular forms (TMF), c.f., [AHS01, AHR10]. Such an index theorem, if it existed, would probe some very subtle analytical and differential geometric aspects of these field theories, mimicking the known intricacies in TMF [Hop02]. In so doing, it would also provide differential-geometric tools for studying powerful homotopy invariants.

This paper offers a glimpse at this picture by putting the geometry of 2-dimensional field theories in direct contact with the string orientation of  $\mathrm{TMF} \otimes \mathbb{C}$  and proving a baby case of this hoped-for index theorem. We work with a version of the differential cocycle model for  $\mathrm{TMF} \otimes \mathbb{C}$  developed in [BE13], wherein functions on a certain super double loop space furnish cocycles. Analyzing two types of classical field theories and their 1-loop quantum partition functions yields three main results: (1) an analytic construction of elliptic Mathai–Quillen Thom forms for  $\mathrm{TMF} \otimes \mathbb{C}$ , (2) an analytic construction of the Witten class of a family of string manifolds, and (3) an index theorem equating the pushforwards associated to these two constructions.

This settles a simplified version of the program initiated by Segal [Seg88, Seg04] and Stolz–Teichner [ST04, ST11] which we paraphrase as follows. Techniques in 2-dimensional field theories construct the string orientation of  $\mathrm{TMF} \otimes \mathbb{C}$ , and there is an index theorem equating analytic and topological pushforwards. The *topological* pushforward comes from

the partition function of a modified free fermion field theory; this constructs a cocycle representative of the elliptic Thom class. The *analytic* pushforward comes from the 1-loop partition function of a families-version of the supersymmetric nonlinear supersymmetric sigma model related to Witten's Dirac operator on loop space [Wit88, Wit99]. These beginnings for an index theory associated with 2-dimensional quantum field theory have obvious enhancements from physics: there is much more structure to a quantum field theory than its partition function. The hope is that these enhancements will continue to encode interesting topology. One possible path is the Stolz–Teichner program; we explain in §1.8 how to view their conjectures as categorifications of the results of this paper.

To the best of our knowledge, there is no clear physical argument for why the analytic and topological pushforwards constructed below agree. Of course, one might have anticipated this in analogy to the situation in K-theory, with the topological pushforward being constructed from Thom classes and analytic pushforward from the index of the Dirac operator. To make this analogy as explicit as possible, we construct these pushforwards in complexified K-theory from a 1-dimensional version of the constructions for  $\mathrm{TMF} \otimes \mathbb{C}$ . This gives a new construction of the Mathai–Quillen form in  $K \otimes \mathbb{C}$  using techniques from path integrals and 1-loop partition functions which might be of independent interest. We emphasize that the resulting equality of pushforwards in  $K \otimes \mathbb{C}$  is distinct from the standard physical proof of the local index theorem [AG83]: the usual argument identifies two calculations for the partition function of supersymmetric quantum mechanics, one in the Hamiltonian and the other in the Lagrangian formulation of the theory. Below we identify the partition functions for two *different* field theories, both computed in the Lagrangian formulation.

In the next subsection we introduce the minimal ingredients necessary to state our main results, and in the remainder of the section we some provide background and outline the key arguments.

**1.1. Statement of results.** The differential cocycle model for  $\mathrm{TMF}(M) \otimes \mathbb{C}$  starts with the *super double loop stack* of  $M$ , denoted  $\mathcal{L}^{2|1}(M)$ . It has objects  $(\Lambda, \phi)$  for  $\Lambda \subset \mathbb{R}^2$  a 2-dimensional lattice defining a *super torus*  $\mathbb{T}_\Lambda^{2|1} := \mathbb{R}^{2|1}/\Lambda$  and  $\phi: \mathbb{T}_\Lambda^{2|1} \rightarrow M$  a smooth map. There is a substack  $\mathcal{L}_0^{2|1}(M) \subset \mathcal{L}^{2|1}(M)$  consisting of those maps  $\phi$  invariant under the precomposition action of (ordinary) translations  $\mathbb{T}_\Lambda^2 = \mathbb{R}^2/\Lambda \subset \mathbb{R}^{2|1}/\Lambda = \mathbb{T}_\Lambda^{2|1}$ . This is the substack of *constant super tori* in  $M$ . A line bundle  $\omega^{1/2}$  on  $\mathcal{L}_0^{2|1}(M)$  comes from a square root of the Hodge bundle on the moduli stack of elliptic curves. It has a type of complex structure on sections; denote holomorphic sections of the  $k^{\mathrm{th}}$  tensor power by  $\mathcal{O}(\mathcal{L}_0^{2|1}(M); \omega^{k/2})$ . To identify sections with cocycles, note that  $\mathrm{TMF} \otimes \mathbb{C}$  is equivalent to ordinary cohomology with values in the graded ring  $\mathrm{MF}$  of weak modular forms.

**Theorem 1.1.** *The assignment  $M \mapsto \mathcal{O}(\mathcal{L}_0^{2|1}(M); \omega^{\bullet/2})$  defines a sheaf of graded algebras on the site of smooth manifolds, and there is an isomorphism of sheaves*

$$\mathcal{O}(\mathcal{L}_0^{2|1}(-); \omega^{\bullet/2}) \xrightarrow{\sim} \bigoplus_{i+j=\bullet} \Omega_{\mathrm{cl}}^i(-; \mathrm{MF}^j)$$

*with closed differential forms valued in weak modular forms. Hence,  $\mathcal{O}(\mathcal{L}_0^{2|1}(M); \omega^{\bullet/2})$  is a differential cocycle model for  $\mathrm{TMF}(M) \otimes \mathbb{C}$  in the sense of Hopkins–Singer [HS05].*

*Remark 1.2.* The above is essentially Theorem 1.1 in [BE13], with a mild repackaging intended to clarify the connection with super double loop spaces.

A geometric family of oriented manifolds  $\pi: M \rightarrow B$  determines a vector bundle  $\mathcal{N}^{2|1}(M/B) \rightarrow \mathcal{L}_0^{2|1}(M)$  whose fiber at  $(\Lambda, \phi)$  is the orthogonal complement of the constant sections in  $\Gamma(\mathbb{T}_\Lambda^{2|1}, \phi^*T(M/B))$  for  $T(M/B)$  the tangent bundle of the fibers. The classical action for the nonlinear supersymmetric sigma model is a function on  $\mathcal{L}^{2|1}(M)$

whose Hessian on  $\mathcal{L}_0^{2|1}(M) \subset \mathcal{L}^{2|1}(M)$  determines a quadratic function on sections

$$(1) \quad \text{Hess}(\sigma) = \int_{\mathbb{T}_\Lambda^{2|1}} \langle \sigma, \nabla_{\partial_z} \nabla_D \sigma \rangle, \quad \sigma \in \Gamma(\mathcal{L}_0^{2|1}(M), \mathcal{N}^{2|1}(M/B))$$

where  $D = \partial_\theta + \theta \partial_{\bar{z}}$  and  $\partial_z$  are complex vector fields on  $\mathbb{T}_\Lambda^{2|1}$ . We'll find that  $\nabla_{\partial_z} \nabla_D$  is determined by a deformation of a Laplacian on tori by the curvature of  $M$ ; to emphasize dependence on  $\pi: M \rightarrow B$  we write  $\Delta_{M/B}^{2|1} := \nabla_{\partial_z} \nabla_D$ . The formalism of 1-loop quantization points towards evaluation of the functional integral on the left hand side

$$(2) \quad \int_{\mathcal{N}^{2|1}(M/B)} e^{-\text{Hess}(\sigma)} d\sigma = \text{sdet}_\zeta(\Delta_{M/B}^{2|1})$$

which we make rigorous via the  $\zeta$ -super determinant on the right hand side. This determinant defines a line bundle  $\text{Str}(M/B)$  with section over  $\mathcal{L}_0^{2|1}(M)$ .

**Theorem 1.3.** *The section*

$$\int_{\mathcal{N}^{2|1}(M/B)} e^{-\text{Hess}(\sigma)} \frac{d\sigma}{Z^{\dim(M) - \dim(X)}} = \text{Wit}(M/B) \in \mathcal{O}(\mathcal{L}_0^{2|1}(X); \text{Str}(M/B))$$

represents the twisted Witten class of the family  $\pi: M \rightarrow B$  as a differential cocycle in  $\text{TMF}^0(M) \otimes \mathbb{C}$ , for  $Z$  a normalization that is essentially the Dedekind  $\eta$ -function.

The line bundle  $\text{Str}(M/B)$  is concordant to the trivial line bundle if and only if the family  $\pi: M \rightarrow B$  has a rational string structure, and a choice of rational string structure  $H \in \Omega^3(M)$  with  $dH = p_1(T(M/B))$  specifies a concordance between  $\text{Wit}(M/B)$  and a function  $\text{Wit}_H(M/B) \in \mathcal{O}(\mathcal{L}_0^{2|1}(M/B))$  representing the modular Witten class in  $\text{TMF}^0(M) \otimes \mathbb{C}$ .

Next we describe the construction of the Thom cocycle. For an oriented real vector bundle  $V \rightarrow M$ , the pullback over itself has a canonical section  $\mathbf{x} \in \Gamma(V, p^*V)$ . Define a vector bundle  $\mathcal{F}^{2|1}(V) \rightarrow \mathcal{L}_0^{2|1}(V)$  whose fiber at  $(\Lambda, \phi)$  is  $\Gamma(\mathbb{T}_\Lambda^{2|1}, \phi^* p^* \text{IV})$  where  $\text{IV}$  is the parity reversal of  $V$ . If we equip  $V$  with a metric and compatible connection, we obtain the function on sections

$$(3) \quad \mathcal{S}_{\text{MQ}}(\Psi) = \int_{\mathbb{T}_\Lambda^{2|1}} \left( \frac{1}{2} \langle \Psi, \nabla_D \Psi \rangle + i \left\langle \frac{\phi^* \mathbf{x}}{\text{vol}^{1/2}}, \Psi \right\rangle \right), \quad \Psi \in \Gamma(\mathcal{L}^{2|1}(M); \mathcal{F}^{2|1}(V)),$$

where  $\langle -, - \rangle$  and  $\nabla$  are the pullback of the metric and connection of  $p^*V$  along  $\phi$ , and  $\text{vol}$  is the volume of the torus  $\mathbb{R}^2/\Lambda$ . This is a 2-dimensional generalization of the classical action studied by Mathai and Quillen [MQ86]. Similar  $\zeta$ -determinant techniques allow one to rigorously define the functional integral involving  $\mathcal{S}_{\text{MQ}}$ ; see (22). The relevant family of operators is denoted  $\mathcal{D}_V^{2|1}$ , and is a deformation of a family of Dirac operators on tori by the curvature of  $V$ . The associated determinant line bundle is denoted  $\text{Str}(V)$ .

**Theorem 1.4.** *The section*

$$\int_{\mathcal{F}^{2|1}(V)} e^{-\mathcal{S}_{\text{MQ}}(\Psi)} \frac{d\Psi}{(2\pi \cdot Z)^{\dim(V)}} = \sigma_{\text{MQ}}(V) \in \Gamma_{\text{cvs}}(\mathcal{L}_0^{2|1}(V); \omega^{\dim(V)/2} \otimes \text{Str}(V))$$

represents the twisted Thom class of  $V$  in  $\text{TMF} \otimes \mathbb{C}$  associated with the complexified string orientation of  $\text{TMF}$ , where  $Z$  is as in Theorem 1.3.

The line bundle  $\text{Str}(V)$  is concordant to the trivial line bundle if and only if  $V$  has a rational string structure. A choice of rational string structure  $H \in \Omega^3(M)$  with  $dH = p_1(V)$  picks out a concordance between  $\sigma_{\text{MQ}}(V)$  and a section  $\sigma_{\text{MQ}}(V, H) \in \Gamma_{\text{cs}}(\mathcal{L}_0^{2|1}(V); \omega^{\dim(V)/2})$  that represents the (untwisted) Thom class in  $\text{TMF}_{\text{cvs}}^{\dim(V)}(V) \otimes \mathbb{C}$ .

Call the differential cocycle  $\sigma_{\text{MQ}}(V, H)$  the *elliptic Mathai–Quillen form*. It determines a differential refinement of the string orientation of  $\text{TMF} \otimes \mathbb{C}$ . For a Riemannian embedding  $M \hookrightarrow \mathbb{R}^N$ , we get an embedding  $i: M \rightarrow \mathbb{R}^N \times B$  with normal bundle  $\nu$ . The Thom isomorphism for  $\nu$  together with the inverse to the suspension isomorphism defines the *topological pushforward*, denoted  $\pi_!^{\text{top}}$ . The *analytic pushforward*, denoted  $\pi_!^{\text{an}}$ , uses the

Witten class for the family  $\pi: M \rightarrow B$  to modify a canonical volume form on the fibers  $\mathcal{L}_0^{2|1}(M) \rightarrow \mathcal{L}_0^{2|1}(B)$  coming from the integration of differential forms.

**Theorem 1.5.** *Let  $\pi: M \rightarrow B$  be a geometric family of oriented manifolds with fiber dimension  $n$ . There is a canonical isomorphism of super determinant line bundles for the families of operators  $\mathcal{D}_V^{2|1}$  and  $\Delta_{M/B}^{2|1}$  over  $\mathcal{L}_0^{2|1}(M)$  compatible with their respective super determinant sections. This implies that the analytic and topological pushforwards on differential cocycles agree for geometric families of rational string manifolds,*

$$\pi_!^{\text{an}} = \pi_!^{\text{top}}: \mathcal{O}(\mathcal{L}_0^{2|1}(M); \omega^{k/2}) \rightarrow \mathcal{O}(\mathcal{L}_0^{2|1}(B); \omega^{(k-n)/2}).$$

As we'll explain shortly, the above theorem is more of a geometric rephrasing of the relevant Riemann–Roch factors in the index theorem over  $\mathbb{C}$  rather than a new proof. However, this rephrasing makes direct contact with the geometry of field theories, pointing towards generalizations in *extended* (functorial) field theories as we shall explain in §1.8.

**1.2. Mathai–Quillen forms.** The vector bundle  $V \rightarrow M$  can be pulled back over itself,

$$\begin{array}{ccc} p^*V & \xrightarrow{\quad} & V \\ \mathbf{x} \left( \begin{array}{c} \uparrow \\ \downarrow \end{array} \right. & & \downarrow \\ V & \xrightarrow{\quad p \quad} & M \end{array}$$

and the pullback has a tautological section  $\mathbf{x}$ . If we equip  $V$  with a metric  $\langle -, - \rangle$  and compatible connection  $\nabla$  with curvature  $F$ , the Mathai–Quillen Thom form in ordinary cohomology is the Berezinian integral (c.f. [BGV92] §1.6)

$$(4) \quad \text{Th}(V) = \frac{1}{(2\pi)^{\dim(V)}} \int \exp \left( -\frac{1}{2} \langle \mathbf{x}, \mathbf{x} \rangle - i \langle \nabla \mathbf{x}, - \rangle - \frac{1}{2} \langle -, F- \rangle \right) \in \Omega_{\text{cl}, \text{cvs}}^\bullet(V)$$

where here and throughout the subscript cvs denotes *compact vertical support* (or rapidly decreasing forms) in the fiber directions of  $V$ , and  $\Omega_{\text{cl}}$  denotes the sheaf of *closed* differential forms. By the Riemann–Roch theorem, the complexification of the K-theoretic Thom class fits into the commuting diagram

$$\begin{array}{ccc} K^\bullet(M) & \xrightarrow{\text{ch}} & \text{HP}^\bullet(M) \\ \sigma(V) \smile \downarrow & & \downarrow \text{Th}(V) \hat{A}(V)^{-1} \smile \\ K_{\text{cvs}}^{\bullet+n}(V) & \xrightarrow{\text{ch}} & \text{HP}_{\text{cvs}}^{\bullet+n}(V) \end{array}$$

where  $\sigma(V)$  is the K-theoretic Thom class of  $V$  associated to the spin orientation,  $\text{HP} \cong K \otimes \mathbb{C}$  is 2-periodic cohomology over  $\mathbb{C}$ , and  $\hat{A}(V)^{-1}$  is the inverse of the  $\hat{A}$ -class of  $V$ , which is a characteristic class associated with the power series

$$\frac{z/2}{\sinh(z/2)} = \exp \left( \sum_{k=1}^{\infty} \frac{z^{2k}}{2k(2\pi i)^{2k}} 2\zeta(2k) \right) \in \mathbb{R}[[z]].$$

Mathai and Quillen constructed a differential form representative of  $\text{Th}(V) \hat{A}(V)^{-1}$  by a careful study of trace maps for the Clifford modules that can be used to construct the class  $\sigma(V)$  in K-theory. Below we offer a different approach, constructing this form from a  $\zeta$ -regularized super determinant of a family of deformed Dirac operators on  $S^1$ . This perspective has a natural generalization to deformed Dirac operators on  $\mathbb{T}^2$ . We get a

differential form refinement of the vertical right arrow in the commutative diagram

$$(5) \quad \begin{array}{ccc} \mathrm{TMF}^\bullet(M) & \xrightarrow{\mathrm{ch}} & \mathrm{H}^\bullet(M; \mathrm{MF}) \\ \sigma(V) \smile \downarrow & & \downarrow \mathrm{Th}(V) \mathrm{Wit}(V)^{-1} \smile \\ \mathrm{TMF}_{\mathrm{cvs}}^{\bullet+n}(V) & \xrightarrow{\mathrm{ch}} & \mathrm{H}_{\mathrm{cvs}}^{\bullet+n}(V; \mathrm{MF}) \end{array}$$

for vector bundles with string structure. In the above,  $\mathrm{H}(-; \mathrm{MF}) \cong \mathrm{TMF}(-) \otimes \mathbb{C}$  is cohomology with coefficients in weak modular forms,  $\sigma(V)$  is the TMF Thom class of  $V$  associated to the string orientation [AHS01, AHR10],  $\mathrm{Th}(V)$  is the Thom class in ordinary cohomology, and  $\mathrm{Wit}(V)$  is the Witten class of  $V$  associated with the power series

$$(e^{z/2} - e^{-z/2}) \prod_{n \geq 1} \frac{(1 - q^n e^{z/2})(1 - q^n e^{-z/2})}{(1 - q^n)^2} = \exp \left( \sum_{k \geq 1} \frac{E_{2k}(q)}{2k(2\pi i)^{2k}} z^{2k} \right) \in \mathbb{C}[[z, q]]$$

where  $E_{2k}$  is the  $2k^{\mathrm{th}}$  Eisenstein series (see §A.4). We will also have use for the class  $\mathrm{Wit}_H(V)$  associated to the power series,

$$(6) \quad \exp \left( \sum_{k \geq 2} \frac{E_{2k}(q)}{2k(2\pi i)^{2k}} z^{2k} \right) \in \mathbb{C}[[z, q]]$$

which we shall apply to vector bundles  $V \rightarrow M$  equipped with a 3-form  $H \in \Omega^3(M)$  satisfying  $dH = p_1(V)$ , i.e., a (geometric) rational string structure. The existence of the 3-form  $H$  means that  $\mathrm{Wit}(V)$  and  $\mathrm{Wit}_H(V)$  are cohomologous; the point is that (6) clearly gives a class with values in modular forms, as the Eisenstein series  $E_{2k}(q)$  are modular forms for  $k \geq 2$ .

**1.3. Analytic and topological pushforwards over  $\mathbb{C}$ .** Let  $\pi: M \rightarrow B$  be family of spin manifolds with fiber dimension  $n$ . Choose an embedding  $M \hookrightarrow \mathbb{R}^N$ , which determines an embedding  $i: M \hookrightarrow B \times \mathbb{R}^N$ . Let  $\nu$  be the normal bundle of  $M$  for  $i$  of dimension  $N - n$ . Then the *topological pushforward* in K-theory sits in the diagram on the left

$$\begin{array}{ccc} \mathrm{K}^\bullet(M) & \xrightarrow{\sigma(\nu) \smile} & \mathrm{K}_{\mathrm{cvs}}^{\bullet+N-n}(B \times \mathbb{R}^M) \\ \pi_!^{\mathrm{top}} \searrow & & \swarrow \Sigma^{-N} \\ & \mathrm{K}^{\bullet-n}(B) & \end{array} \quad \begin{array}{ccc} \mathrm{HP}^\bullet(M) & \xrightarrow{\mathrm{Th}(\nu) \hat{A}(\nu)^{-1} \smile} & \mathrm{HP}_{\mathrm{cvs}}^{\bullet+N-n}(B \times \mathbb{R}^M) \\ \pi_!^{\mathrm{top}} \searrow & & \swarrow \int_{B \times \mathbb{R}^N / B} \\ & \mathrm{HP}^{\bullet-n}(B) & \end{array}$$

where  $\Sigma^{-N}$  is the inverse to the suspension isomorphism. The diagram on the right is the complexification of the one on the left, defining the topological pushforward in complexified K-theory. In this case, the inverse to the suspension isomorphism is simply the integration over the fibers of  $B \times \mathbb{R}^N \rightarrow B$ .

The string orientation of TMF gives a completely analogous story for  $\pi: M \rightarrow B$  a family of string manifolds. We get

$$\begin{array}{ccc} \mathrm{TMF}^\bullet(M) & \xrightarrow{\sigma(\nu) \smile} & \mathrm{TMF}_{\mathrm{cvs}}^{\bullet+N-n}(B \times \mathbb{R}^M) \\ \pi_!^{\mathrm{top}} \searrow & & \swarrow \Sigma^{-N} \\ & \mathrm{TMF}^{\bullet-n}(B) & \end{array} \quad \begin{array}{ccc} \mathrm{H}^\bullet(M; \mathrm{MF}) & \xrightarrow{\mathrm{Th}(\nu) \mathrm{Wit}(\nu)^{-1} \smile} & \mathrm{H}_{\mathrm{cvs}}^{\bullet+N-n}(B \times \mathbb{R}^M; \mathrm{MF}) \\ \pi_!^{\mathrm{top}} \searrow & & \swarrow \int_{B \times \mathbb{R}^N / B} \\ & \mathrm{H}^{\bullet-n}(B; \mathrm{MF}) & \end{array}$$

and this defines the topological pushforward and its complexification.

The *analytic pushforward* in K-theory (i.e., index of the Dirac operator) gives a wrong way map  $\pi_!^{\text{an}}$ , and the Riemann–Roch theorem states that the diagram commutes:

$$(7) \quad \begin{array}{ccc} K^\bullet(M) & \xrightarrow{\text{ch}} & HP^\bullet(M) \\ \pi_!^{\text{an}} \downarrow & & \downarrow \int_{M/B} - \smile \hat{A}(M/B) \\ K^{\bullet-n}(B) & \xrightarrow{\text{ch}} & HP^{\bullet-n}(B), \end{array}$$

where  $\hat{A}(M/B)$  is the  $\hat{A}$ -form of the vertical tangent bundle of  $\pi$ . The families index theorem states the the topological and analytic pushforwards agree,  $\pi_!^{\text{top}} = \pi_!^{\text{an}}$ .

Witten’s physical reasoning [Wit99] leads one to hope for a TMF-generalization of the Dirac operator and analytic pushforward. Although it remains a difficult and open problem to construct such an operator (often referred to as *the Dirac operator on loop space*), for  $\pi: M \rightarrow B$  a family with fiberwise string structures, any candidate analytic pushforward should sit in the diagram

$$(8) \quad \begin{array}{ccc} \text{TMF}^\bullet(X) & \xrightarrow{\text{ch}} & H^\bullet(X; \text{MF}) \\ \pi_!^{\text{an}} \downarrow \text{---} & & \downarrow \int_{M/B} - \smile \text{Wit}(M/B) \\ \text{TMF}^{\bullet-n}(M) & \xrightarrow{\text{ch}} & H^{\bullet-n}(M; \text{MF}). \end{array}$$

Below we use analytic techniques to construct a differential cocycle refinement for the vertical arrow on the right via a 1-loop quantization procedure for the supersymmetric nonlinear sigma model with target  $M$ . This is the same physical theory studied by Witten in his construction of the Witten genus.

In spite of the absence of an analytic pushforward in TMF, there is an index theorem of sorts for the pushforwards in  $\text{TMF} \otimes \mathbb{C}$ , which amounts to commutativity of the diagram

$$\begin{array}{ccc} H^\bullet(M; \text{MF}) & \xrightarrow{\text{Th}(\nu)\text{Wit}(\nu)^{-1} \smile} & H_{\text{cvs}}^{\bullet+N-n}(B \times \mathbb{R}^M; \text{MF}) \\ \int_{M/B} - \smile \text{Wit}(M/B) \searrow & & \swarrow \int_{B \times \mathbb{R}^N / B} \\ & H^{\bullet-n}(B; \text{MF}). \end{array}$$

Furthermore, given a differential cocycle model it makes sense to ask for a differential refinement of this diagram. This isn’t terribly deep: it is equivalent to refining

$$(9) \quad \text{Wit}(\nu) = \text{Wit}(M/B),$$

to the level of differential cocycles. Any differential cocycle model is isomorphic to the differential form model for  $\text{TMF} \otimes \mathbb{C}$ , so (9) can always be rephrased as an equality of differential forms. Such an equality is not hard to cook up; in our case all one needs is for the embedding  $M \hookrightarrow \mathbb{R}^N$  to be compatible with the Riemannian structure on  $M$  that defines the Pontryagin forms. The more interesting part of the story (in our view) is in the analytic construction of the differential forms themselves. This comes from analysis of families of operators over certain super stacks we review presently.

**Notation 1.6.** Hereafter we will use the notation  $\hat{A}(V)^{-1}$ ,  $\hat{A}(M/B)$ ,  $\text{Wit}(V)^{-1}$ ,  $\text{Wit}_H(V)^{-1}$ ,  $\text{Wit}(M/B)$ , and  $\text{Wit}_H(M/B)$  to denote differential refinements of the classes  $[\hat{A}(V)^{-1}]$ ,  $[\hat{A}(M/B)]$ ,  $[\text{Wit}(V)^{-1}]$ ,  $[\text{Wit}_H(V)^{-1}]$ ,  $[\text{Wit}(M/B)]$ , and  $[\text{Wit}_H(M/B)]$ . Such refinements typically depend on choices of metric and connection that define the Pontryagin forms representing the Pontryagin classes.

**1.4. A brief description of the differential cocycle models.** To emphasize certain structural aspects of the story, we will prove analogs of Theorems 1.1–1.5 for de Rham cohomology and K-theory with complex coefficients. In this subsection we quickly review the relevant cocycle models with details in §2.1, §3.1 and §4.1.

For  $d = 0, 1, 2$  and  $M$  a smooth manifold, define a stack  $\mathcal{L}^{d|1}(M)$  whose objects<sup>1</sup> are super tori with a map to  $M$ ,  $\phi: \mathbb{R}^{d|1}/\mathbb{Z}^d \rightarrow M$  and whose morphisms are isometries  $\mathbb{R}^{d|1}/\mathbb{Z}^d \xrightarrow{\sim} \mathbb{R}^{d|1}/\mathbb{Z}^d$  compatible with the maps to  $M$ . The stack of *constant  $d|1$ -dimensional super tori* is the full substack  $\mathcal{L}_0^{d|1}(M) \subset \mathcal{L}^{d|1}(M)$ , for which  $\phi: \mathbb{R}^{d|1}/\mathbb{Z}^d \rightarrow M$  is invariant under the pre-composition action of  $\mathbb{R}^d/\mathbb{Z}^d \subset \mathbb{R}^{d|1}/\mathbb{Z}^d$  by translation. For each  $d$  there are line bundles  $\omega^{1/2}$  over  $\mathcal{L}_0^{d|1}(M)$ , and the assignment  $M \mapsto \Gamma(\mathcal{L}_0^{d|1}(M); \omega^{\bullet/2})$  defines a sheaf on the site of smooth manifolds. There are isomorphisms of sheaves

$$\begin{aligned} (10) \quad \Gamma(\mathcal{L}_0^{0|1}(-), \omega^{\bullet/2}) &= \Omega_{\text{cl}}^{\bullet}(-) \\ (11) \quad \Gamma(\mathcal{L}_0^{1|1}(-), \omega^{\bullet/2}) &\cong \begin{cases} \bigoplus_i \Omega_{\text{cl}}^{2i}(-) & \bullet = \text{even} \\ \bigoplus_i \Omega_{\text{cl}}^{2i+1}(-) & \bullet = \text{odd} \end{cases} \\ (12) \quad \mathcal{O}(\mathcal{L}_0^{2|1}(-), \omega^{\bullet/2}) &\cong \bigoplus_{i+j=\bullet} \Omega_{\text{cl}}^i(-) \otimes \text{MF}^j \end{aligned}$$

using the work of [HKST11] and [BE13]. This gives differential cocycle models for cohomology with  $\mathbb{C}$ -coefficient,  $K \otimes \mathbb{C}$  and  $\text{TMF} \otimes \mathbb{C}$ , respectively.

The computations leading to the above isomorphisms make use of preferred atlases for the stacks  $\mathcal{L}_0^{d|1}(M)$  given by

$$\begin{aligned} (13) \quad u: \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) &\twoheadrightarrow \mathcal{L}_0^{0|1}(M) \\ (14) \quad u: \mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) &\twoheadrightarrow \mathcal{L}_0^{1|1}(M) \\ (15) \quad u: L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) &\twoheadrightarrow \mathcal{L}_0^{2|1}(M) \end{aligned}$$

where  $L = \{(\ell_1, \ell_2) \in \mathbb{C}^2 \mid \ell_1/\ell_2 \in \mathfrak{h}\}$  is the *space of lattices in  $\mathbb{C}$*  (see §A.4). Indeed, such covers come from identifying a super circle or super torus with a quotient  $\mathbb{R}^{1|1}/r\mathbb{Z}$  for  $r \in \mathbb{R}_{>0}$  or  $\mathbb{R}^{2|1}/\ell_1\mathbb{Z} \oplus \ell_2\mathbb{Z}$ , and a map  $\phi: \mathbb{R}^{d|1}/\mathbb{Z}^d \rightarrow M$  invariant under the  $\mathbb{R}^d/\mathbb{Z}^d$ -action with a map  $\phi_0: \mathbb{R}^{0|1} \rightarrow M$ , using that  $(\mathbb{R}^{d|1}/\mathbb{Z}^d)/(\mathbb{R}^d/\mathbb{Z}^d) \cong \mathbb{R}^{0|1}$ . The isomorphism  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \cong \Pi TM$  shows that functions on the stacks  $\mathcal{L}_0^{0|1}(M)$ ,  $\mathcal{L}_0^{1|1}(M)$  and  $\mathcal{L}_0^{2|1}(M)$  are differential forms on  $M$  with values in  $\mathbb{C}$ ,  $C^\infty(\mathbb{R}_{>0})$  and  $C^\infty(L)$  (for  $d = 0, 1, 2$ ) that are invariant under isomorphisms in the relevant stack.

**1.5. Mathai–Quillen Thom forms from families of operators over  $\mathcal{L}_0^{d|1}(M)$ .** For  $p: V \rightarrow M$  a real vector bundle and  $d = 0, 1, 2$ , define a vector bundle  $\mathcal{F}^{d|1}(V) \rightarrow \mathcal{L}_0^{d|1}(M)$  whose fiber at a map  $\phi: \mathbb{R}^{d|1}/\mathbb{Z}^d \rightarrow V$  is  $\Gamma(\mathbb{R}^{d|1}/\mathbb{Z}^d, \Pi(\phi^*p^*V))$ , where  $\Pi$  denotes the parity reversal functor. This bundle is of infinite rank for  $d > 0$ . Define a functional on sections

$$(16) \quad \mathcal{S}_{\text{MQ}}(\Psi) = \int_{\mathbb{R}^{d|1}/\mathbb{Z}^d} \left( \frac{1}{2} \langle \Psi, \nabla_D \Psi \rangle - i \left\langle \Psi, \frac{\phi^* \mathbf{x}}{\text{vol}^{1/2}} \right\rangle \right), \quad \Psi \in \Gamma(\mathbb{R}^{d|1}/\mathbb{Z}^d, \phi^*p^*\Pi V)$$

for  $\text{vol}$  is the volume of  $\mathbb{R}^d/\mathbb{Z}^d$ ,

$$D = \begin{cases} \partial_\theta & d = 0 \\ \partial_\theta - i\theta\partial_t & d = 1 \\ \partial_\theta + \theta\partial_{\bar{z}} & d = 2 \end{cases}$$

is an odd vector field on  $\mathbb{R}^{d|1}$ , and we use the Berezinian measure on  $\mathbb{R}^{d|1}/\mathbb{Z}^d$  determined by a volume form on  $\mathbb{R}^d/\mathbb{Z}^d$ . This action is essentially a super-space (or worldsheet) version of the one studied by Mathai and Quillen; see also [Wu05] for this worldsheet point of view. A consequence of Mathai and Quillen’s work is the following.

<sup>1</sup>For simplicity, we omit family parameters (i.e.,  $S$ -points) throughout the introduction.

**Proposition 1.7.** *The (finite-dimensional) Berezin integral*

$$\mathrm{Th}(V) = \frac{1}{(2\pi)^{\dim(V)}} \int_{\mathcal{F}^{0|1}(V)/\mathcal{L}^{0|1}(V)} e^{-\mathcal{S}_{\mathrm{MQ}}(\Psi)} d\Psi \in \Gamma_{\mathrm{cs}}(\mathcal{L}^{0|1}(V); \omega^{\dim(V)/2}) \cong \Omega_{\mathrm{cl}, \mathrm{cs}}^{\dim(V)}(V)$$

*constructs the Mathai–Quillen Thom form in de Rham cohomology.*

Although the calculation above is well-trodden terrain, we re-prove the result in §2 to set the stage for the generalization when  $d = 1, 2$ . The generalization uses  $\zeta$ -regularization techniques inspired by physics to define the infinite-dimensional analog of the integral in Proposition 1.7; we explain this physical motivation in §1.7.

One feature we exploit is the action of  $\mathbb{R}^d/\mathbb{Z}^d$  on sections of  $\mathcal{F}(V)$ . This induces gradings on the pull back  $\mathcal{F}^{d|1}(V)$  along  $p$  to the atlases (14) and (15) resulting in decompositions

$$(17) \quad u^* \mathcal{F}^{1|1}(V) \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n^{1|1}(V), \quad u^* \mathcal{F}^{2|1}(V) \cong \bigoplus_{(n,m) \in \mathbb{Z}^2} \mathcal{F}_{n,m}^{2|1}(V)$$

where the index  $n$  or  $(n, m)$  corresponds to the weight of the torus action. Each of these weight spaces is finite-dimensional, so the integral of  $\exp(-\mathcal{S}_{\mathrm{MQ}}(\Psi))$  is well-defined on each restriction. There are a few ways to regularize the product of these finite-dimensional contributions; we take one that connects directly with the Riemann–Roch factors  $\hat{A}(V)^{-1}$  and  $\mathrm{Wit}(V)^{-1}$ .

Let  $\mathcal{F}_0^{2|1}(V) := \mathcal{F}_{0,0}^{2|1}(V)$ , and  $\mathcal{F}_\perp^{d|1}(V)$  denote the orthogonal complement of  $\mathcal{F}_0^{d|1}(V)$  in  $u^* \mathcal{F}^{d|1}(V)$ . We will study the restriction of  $\mathcal{S}_{\mathrm{MQ}}$  to  $\mathcal{F}_0^{d|1}(V)$  and  $\mathcal{F}_\perp^{d|1}(V)$ .

**Lemma 1.8.** *The Berezinian integral of the restriction of  $\exp(-\mathcal{S}_{\mathrm{MQ}})$  to sections of  $\mathcal{F}_0^{d|1}(V)$  produces a function on  $\mathcal{L}_0^{d|1}(V)$  that equals the image of the Mathai–Quillen form in de Rham cohomology (up to a factor of  $2\pi^{-\dim(V)}$ ) using the maps (11) and (12).*

Let  $\mathcal{D}_V^{d|1}$  denote the restriction of  $\nabla_D$  to  $\mathcal{F}_\perp^{d|1}(V)$ ; we use this notation both to emphasize the dependence of this operator on  $V$  and  $d$ , and because of its standing as a deformed Dirac operator (as we shall see shortly). We find that the restriction of  $\mathcal{S}_{\mathrm{MQ}}$  is

$$(18) \quad \mathcal{S}_{\mathrm{MQ}}(\Psi) = \int_{\mathbb{R}^{d|1}/\mathbb{Z}^d} \langle \Psi, \mathcal{D}_V^{d|1} \Psi \rangle, \quad \Psi \in \Gamma(\mathcal{F}_\perp^{d|1}(V)).$$

This follows essentially from the fact that the  $L^2$ -inner product of a zero Fourier mode with a nonzero Fourier mode vanishes, so the integral of  $\langle \Psi, \phi^* \mathbf{x} \rangle$  is zero (since  $\phi^* \mathbf{x}$  is a constant section); see Lemmas 3.10 and 4.10 for details. We also find

$$(19) \quad \mathcal{D}_V^{1|1} = \begin{cases} \mathrm{id} & \text{on even sections} \\ i\nabla_{\partial_t} + F & \text{on odd sections} \end{cases}$$

$$(20) \quad \mathcal{D}_V^{2|1} = \begin{cases} \mathrm{id} & \text{on even sections} \\ i\nabla_{\partial_{\bar{z}}} + F & \text{on odd sections} \end{cases}$$

where  $F$  is an order zero differential operator on  $\mathcal{F}_\perp^{d|1}(V)$  coming from the curvature 2-form of  $V$ . By the usual yoga for Gaussian integrals in infinite-dimensions (see §A.5),

$$\int_{\mathcal{F}_\perp^{d|1}(V)} \exp(-\mathcal{S}_{\mathrm{MQ}}) d\Psi = \mathrm{sdet}_\zeta(\mathcal{D}_V^{d|1})$$

and so invoking Fubini’s theorem formally we define

$$(21) \quad \int_{\mathcal{F}^{d|1}(V)} \exp(-\mathcal{S}_{\mathrm{MQ}}) d\Psi := \mathrm{sdet}_\zeta(\mathcal{D}_V^{d|1}) \cdot \int_{\mathcal{F}_0^{d|1}(V)} \exp(-\mathcal{S}_{\mathrm{MQ}}(\Psi)) d\Psi.$$

We’ll find that the  $\zeta$ -super determinant provides the correct Riemann–Roch factor (up to a normalization) that mediates between the de Rham Thom class and the K-theoretic Mathai–Quillen Thom form. This gives a K-theory analog of Theorem 1.4.



**Proposition 1.9.** *The section*

$$\int_{\mathcal{F}^{1|1}(V)} \exp(-\mathcal{S}_{\text{MQ}}) d\Psi \frac{d\Psi}{(2\pi \cdot Z)^{\dim(V)}} = \text{Th}(V) \cdot \hat{A}(V)^{-1} \in \Gamma_{\text{cs}}(\mathcal{L}_0^{1|1}(V); \omega^{\dim(V)/2})$$

*coincides with the Mathai–Quillen K-theoretic Thom form as a differential cocycle in complexified K-theory, where  $Z$  is the unique normalization factor so that  $\frac{\text{sdet}(\mathcal{D}_{\mathbb{R}}^{1|1})}{Z} = 1$  for  $\mathbb{R}$  the trivial 1-dimensional bundle.*

Some subtleties emerge for (21) in the generalization to  $d = 2$  that have both analytic and topological significance. On the topological side, rational string obstructions prevent the existence of Thom isomorphisms for arbitrary oriented vector bundles. Analytically, the  $\zeta$ -super determinant of  $\mathcal{D}_V^{2|1}$  is only conditionally convergent. We use the grading (17) to fix an order of summation for the associated  $\zeta$ -function, and this defines a renormalized super determinant,  $\text{sdet}_{\zeta}^{\text{ren}}(\mathcal{D}_V^{2|1})$ . As the grading is not invariant under isometries of tori that exchange the pair of circles,  $S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$ ,  $\text{sdet}_{\zeta}^{\text{ren}}(\mathcal{D}_V^{2|1})$  typically fails to be a function on  $\mathcal{L}_0^{2|1}(V)$ . This interacts with the topology:  $\text{sdet}_{\zeta}^{\text{ren}}(\mathcal{D}_V^{2|1})$  defines a section of a line bundle  $\text{Str}(V)$  on  $\mathcal{L}_0^{2|1}(V)$  whose trivializations parametrize rational string structures on  $V$ . In summary, we have

$$(22) \quad \int_{\mathcal{F}^{2|1}(V)} e^{-\mathcal{S}_{\text{MQ}}(\Psi)} d\Psi := \text{sdet}_{\zeta}^{\text{ren}}(\mathcal{D}_V^{2|1}) \cdot \int_{\mathcal{F}_0^{d|1}(V)} \exp(-\mathcal{S}_{\text{MQ}}(\Psi)) d\Psi.$$

which sketches the construction of the elliptic Mathai–Quillen form in Theorem 1.4.

**1.6. Analytic pushforwards from families of operators on  $\mathcal{L}_0^{d|1}(X)$ .** For  $\pi: M \rightarrow B$  a geometric family (see §A.3), define a vector bundle  $\mathcal{N}^{d|1}(M/B)$  over  $\mathcal{L}_0^{d|1}(X)$  whose sections at  $\phi: \mathbb{R}^{d|1}/\mathbb{Z}^d \rightarrow X$  are the orthogonal complement to the constant sections in  $\Gamma(\mathbb{R}^{d|1}/\mathbb{Z}^d; \phi^*T(M/B))$ . We will define an operator on these sections associated to the functional

$$(23) \quad \text{Hess}^{1|1}(\sigma) := -i \int_{S \times_r \mathbb{R}^{1|1}/S} \langle \sigma, \nabla_{\partial_t} \nabla_D \sigma \rangle dt d\theta, \quad \sigma \in \Gamma_0(S \times_r \mathbb{R}^{1|1}, \phi^*T(M/B))$$

when  $d = 1$  and (1) when  $d = 2$ . This is the Hessian of the classical action of the supersymmetric nonlinear sigma model with target  $M$ ; see §1.7. To emphasize the dependence of these operators on  $\pi: M \rightarrow B$ , we write  $\nabla_{\partial_t} \nabla_D = \Delta_{M/B}^{1|1}$  and  $\nabla_{\partial_z} \nabla_D = \Delta_{M/B}^{2|1}$ . We also use this notation because these are essentially first-order deformations of Laplacians

$$(24) \quad \Delta_{M/B}^{1|1} = \begin{cases} (\nabla_{\partial_t})^2 + F \circ \nabla_{\partial_t} & \text{on even sections} \\ \nabla_{\partial_t} & \text{on odd sections} \end{cases}$$

$$(25) \quad \Delta_{M/B}^{2|1} = \begin{cases} \nabla_{\partial_z} \nabla_{\partial_{\bar{z}}} + F \circ \nabla_{\partial_z} & \text{on even sections} \\ \nabla_{\partial_z} & \text{on odd sections.} \end{cases}$$

**Proposition 1.10.** *The function*

$$\frac{\text{sdet}_{\zeta}(\Delta_{M/B}^{1|1})}{Z^{\dim(M) - \dim(X)}} = \hat{A}(M/B) \in C^{\infty}(\mathcal{L}_0^{1|1}(X))$$

*coincides with the  $\hat{A}$ -form for the family  $\pi: M \rightarrow B$  as a differential cocycle in complexified K-theory, where  $Z$  is the normalization factor in Proposition 1.9.*

When  $d = 2$ , we run into the same issues of conditional convergence of the  $\zeta$ -regularized super determinant as in our construction of the elliptic Mathai–Quillen form. As before, we use the action of  $\mathbb{R}^2/\mathbb{Z}^2$  to endow sections of  $\mathcal{N}^{2|1}(M/B)$  with a grading,

$$(26) \quad u^* \mathcal{N}^{2|1}(M/B) \cong \bigoplus_{(n,m) \in \mathbb{Z}_*^2} \mathcal{N}_{n,m}^{2|1}(M/B)$$

where  $\mathbb{Z}_*^2 := \mathbb{Z}^2 \setminus (0, 0)$ . We use this grading to order the sum for the  $\zeta$ -function associated to the operator  $\Delta_{M/B}^{2|1}$ . Denote the associated  $\zeta$ -regularized super determinant by  $\text{sdet}_\zeta^{\text{ren}}(\Delta_{M/B}^{2|1})$ . Then we define

$$(27) \quad \int_{\mathcal{N}^{2|1}(M/B)} e^{-\text{Hess}(\sigma)} d\sigma := \text{sdet}_\zeta^{\text{ren}}(\Delta_{M/B}^{2|1}),$$

which sketches the central object in Theorem 1.3.

**1.7. Physical motivation from path integrals and 1-loop quantization.** There are two classical field theories that lead to our construction of the families Witten class and elliptic Mathai–Quillen form. The first is the *supersymmetric nonlinear sigma model* [Fre99, DEF<sup>+</sup>99] whose fields are maps  $\phi: \mathbb{T}_\Lambda^{2|1} \rightarrow M$  and classical action is

$$\mathcal{S}_\sigma(\phi) = \int_{\mathbb{T}_\Lambda^{2|1}} \langle D\phi, \partial_z \phi \rangle.$$

This function and its differential vanish on  $\mathcal{L}_0^{2|1}(M)$ . The Hessian therefore defines a bilinear form on restriction of the tangent bundle of  $\mathcal{L}^{2|1}(M)$  to  $\mathcal{L}_0^{2|1}(M)$ . Explicitly, the fiber of this tangent bundle at  $(\Lambda, \phi)$  is  $\Gamma(\mathbb{T}_\Lambda^{2|1}, \phi^* TM)$ . The formula for this Hessian was computed in [BE13] §5, leading to formula (1). The  $\zeta$ -super determinant of the operator associated with this Hessian is the 1-loop quantization of the supersymmetric sigma model, in that it reads off the contribution to the partition function from 1-loop Feynman diagrams.

The Riemannian exponential map on  $M$  allows one to identify sections of this tangent bundle with maps  $\phi: \mathbb{T}_\Lambda^{2|1} \rightarrow M$  that are close to maps in  $\mathcal{L}_0^{2|1}(M)$ , i.e., we get a tubular neighborhood of the inclusion. The restriction to sections  $\Gamma(\mathbb{T}_\Lambda^{2|1}, \phi^* T(M/B)) \subset \Gamma(\mathbb{T}_\Lambda^{2|1}, \phi^* TM)$  corresponds to the considering those sections that exponentiate to maps into the fibers of  $\pi: M \rightarrow B$ . Hence, the determinant of the operator associated with this restricted Hessian reads off the contribution to the partition function in each of these fibers. This family of 1-loop partition functions on the fibers (appropriately normalized) is our construction of the Witten class.

The elliptic Mathai–Quillen form comes from successive elaborations on the free fermion field theory. This theory has as fields maps  $\mathbb{T}_\Lambda^{2|1} \rightarrow \Pi V$  for  $V$  a vector space and action

$$\mathcal{S}_{\text{Fer}}(\Psi) = \int_{\mathbb{T}^{2|1}} \langle \Psi, D\Psi \rangle, \quad \Psi: \mathbb{T}_\Lambda^{2|1} \rightarrow \Pi V.$$

There is an evident generalization of this to families over  $\mathcal{L}_0^{2|1}(M)$  for  $V \rightarrow M$  a vector bundle with connection. Then for each  $\phi: \mathbb{T}_\Lambda^{2|1} \rightarrow M$ , we take fields to be sections  $\Gamma(\mathbb{T}_\Lambda^{2|1}, \phi^* \Pi V)$  and classical action

$$(28) \quad \mathcal{S}_{\text{Fer}, \phi}(\Psi) = \int_{\mathbb{T}^{2|1}} \langle \Psi, \nabla_D \Psi \rangle, \quad \Psi \in \Gamma(\mathbb{T}_\Lambda^{2|1}, \phi^* V).$$

If we also choose a section  $v \in \Gamma(M, V)$ , we can add a source term to the action functional above, with the universal case the Mathai–Quillen classical action (3). For each  $\phi: \mathbb{T}_\Lambda^{2|1} \rightarrow M$ , the partition function gotten from fiberwise quantization is the elliptic Mathai–Quillen form.

The quantization procedures above are quantizations of families of free field theories: for each  $\phi: \mathbb{T}_\Lambda^{2|1} \rightarrow M$  or  $\phi: \mathbb{T}_\Lambda^{2|1} \rightarrow V$ , the space of fields is linear and the action is purely quadratic. Hence, quantization is unobstructed, being controlled by a  $\zeta$ -super determinant. However, in families and in the presence of symmetries, Quillen [Qui85] taught us that determinants are no longer numbers, but rather sections of line bundles. Nontriviality of these line bundles (which in our case is a purely stacky phenomenon) is called an *anomaly* in physics. Trivializations of this anomaly in our case are equivalent to geometric rational string structures.

**1.8. Free fermions, Segal–Stolz–Teichner elliptic objects and categorification.** In Stolz and Teichner’s first paper on elliptic cohomology [ST04], they sketched (following Segal [Seg04]) the construction of a fully extended 2-dimensional spin field theory as the quantization of the classical free fermions in a vector bundle  $V \rightarrow M$ . This determines a projective functor  $2\text{-Bord}(M) \rightarrow \mathcal{C}$  where  $2\text{-Bord}(M)$  is a bordism 2-category whose bordisms are equipped with conformal structure, spin structure, and a map to a smooth manifold  $M$ . The category  $\mathcal{C}$  consists of von Neumann algebras, bimodules and bimodules maps. They argue that an appropriate supersymmetric extension of this construction is a candidate geometric cocycle for the TMF Euler class of  $V$ .

More recently, Stolz and Teichner have defined a super Euclidean bordism category over  $M$ , denoted  $2|1\text{-EB}(M)$ , in which one might hope to make sense out of such a supersymmetric extension. Furthermore, they conjecture that for an appropriate fully-extended version of  $2|1\text{-EB}(M)$  (whose ultimate definition is still under investigation), there is a natural bijection

$$(29) \quad \text{TMF}(M) \cong \mathbb{P}\text{Fun}^\otimes(2|1\text{-EB}(M), \mathcal{C}) / \sim \quad (\text{conjectural!})$$

between  $\text{TMF}(M)$  and projective functors to  $\mathcal{C}$ . Below we prove a lower-categorical result: restricting the free fermion construction to *closed* bordisms constructs the Euler class of  $V$  in  $\text{TMF}(M) \otimes \mathbb{C}$ . This is an example of *decategorification*: rather than vector spaces and linear maps associated to bordisms, below we study functions on moduli spaces of tori. Indeed, there is a restriction map coming from the functor  $\mathcal{L}_0^{2|1}(M) \hookrightarrow 2|1\text{-EB}(M)$ ,

$$\text{Fun}^\otimes(2|1\text{-EB}(M), \mathcal{C}) \rightarrow C^\infty(\mathcal{L}_0^{2|1}(M))$$

that evaluates a given field theory on tori, viewed as bordisms from the empty set to itself. The classical field theory defined by the action (28) is indeed a supersymmetric extension of the free fermion theory. Furthermore, we identify the correct enhancement of this theory to also obtain Thom classes.

One might try to refine this construction in  $\text{TMF} \otimes \mathbb{C}$  along the lines of Stolz and Teichner’s original proposal in [ST04]. This requires that one extend our Thom cocycles down, i.e., categorify the functions on the moduli space  $\mathcal{L}_0^{2|1}(M)$  to functors out of a supersymmetric bordism category like  $2|1\text{-EB}(M)$ . The first step in this categorification consists of Fock space quantizations of the free fermions and the supersymmetric nonlinear sigma model, which have plenty of precedent in the physics literature.

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## 2. WARM-UP: $0|1$ -DIMENSIONAL FIELD THEORIES, THOM FORMS AND PUSHFORWARDS

In this section we translate standard objects in de Rham cohomology into the language of  $0|1$ -dimensional field theories. This culminates in Proposition 1.7, which is really a formal reworking of Mathai and Quillen’s original construction; see also Berline, Getzler and Vergne [BGV92] §1.6. The only wrinkle is that we repackage differential forms on  $M$  as functions on a moduli stack of maps from the odd line  $\mathbb{R}^{0|1}$  to  $M$ , and introduce some attendant field-theoretic language due to Hohnhold, Kreck, Stolz and Teichner [HKST11].

**2.1. Fields for the  $0|1$ -dimensional sigma model and de Rham cohomology.** Viewing de Rham cohomology as a  $0|1$ -dimensional field theory was explained in [HKST11]; we overview their results. See §A.2 for a review of model (super) geometries.

**Definition 2.1.** The  $0|1$ -dimensional conformal model geometry has  $\mathbb{R}^{0|1}$  as its model space and the super group  $\mathbb{E}^{0|1} \rtimes \mathbb{R}^\times$  as isometry group, where  $\mathbb{E}^{0|1}$  is  $\mathbb{R}^{0|1}$  as a supermanifold with group structure gotten from viewing  $\mathbb{R}^{0|1}$  as a vector space. The semidirect product  $\mathbb{E}^{0|1} \rtimes \mathbb{R}^\times$  comes from the action  $\theta \mapsto \mu \cdot \theta$ , for  $\theta \in \mathbb{E}^{0|1}(S)$  and  $\mu \in \mathbb{R}^\times(S)$ . We take the

obvious left action of  $\mathbb{E}^{0|1} \rtimes \mathbb{R}^\times$  on  $\mathbb{R}^{0|1}$ . The Lie algebra of  $\mathbb{E}^{0|1}$  has an odd generator  $D$  satisfying  $[D, D] = 2D^2 = 0$ .

**Definition 2.2.** The *fields for the 0|1-dimensional sigma model with target  $M$* , denoted  $\mathcal{L}^{0|1}(M)$ , is the stack whose objects over  $S$  are maps  $\phi: S \times \mathbb{R}^{0|1} \rightarrow M$  and whose morphisms are commuting triangles,

$$(30) \quad \begin{array}{ccc} S \times \mathbb{R}^{0|1} & \xrightarrow{\cong} & S \times \mathbb{R}^{0|1} \\ & \searrow \phi & \swarrow \phi' \\ & M & \end{array}$$

where the horizontal arrow is an  $S$ -family of super conformal isometries.

Define a morphism of stacks  $\mathcal{L}^{0|1}(\text{pt}) \rightarrow \text{pt} // \mathbb{R}^\times$  whose value on objects is constant to  $\text{pt}$  and whose value on a morphism associated to an isometry  $S \times \mathbb{R}^{0|1} \rightarrow S \times \mathbb{R}^{0|1}$  is the map  $S \rightarrow \mathbb{R}^\times$  associated with the dilation of  $\mathbb{R}^{0|1}$ . There is a canonical odd complex line bundle over  $\text{pt} // \mathbb{R}^\times$  associated to the homomorphism

$$(31) \quad \mathbb{R}^\times \subset \mathbb{C}^\times \cong \text{GL}(\mathbb{C}^{0|1}).$$

**Definition 2.3.** Let  $\omega^{k/2}$  denote the line bundle over  $\mathcal{L}^{0|1}(M)$  that is the pullback of the  $k$ th tensor power of the canonical odd line bundle over  $\text{pt} // \mathbb{R}^\times$  along the composition,

$$\mathcal{L}^{0|1}(M) \rightarrow \mathcal{L}^{0|1}(\text{pt}) \rightarrow \text{pt} // \mathbb{R}^\times$$

where the first arrow is induced by  $M \rightarrow \text{pt}$ .

**Proposition 2.4** ([HKST11]). *There is an isomorphism of sheaves of graded algebras,*

$$\Omega_{\text{cl}}^\bullet(-) \xrightarrow{\sim} \Gamma(\mathcal{L}^{0|1}(-); \omega^{\bullet/2}),$$

so that the assignment  $M \mapsto \Gamma(\mathcal{L}^{0|1}(M); \omega^{\bullet/2})$  is a model for de Rham cohomology.

**2.2. An atlas for  $\mathcal{L}^{0|1}(M)$  and its groupoid presentation.** To prove Proposition 2.4, we determine a groupoid presentation of  $\mathcal{L}^{0|1}(M)$  on which we can compute sections of  $\omega^{k/2}$ . There is an evident atlas  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \rightarrow \mathcal{L}^{0|1}(M)$ , on which the map  $\phi$  is evaluation,

$$\text{ev}: \mathbb{R}^{0|1} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \rightarrow M.$$

As always, an atlas has an associated Lie groupoid presentation. Since an  $S$ -family of super conformal isometries is an  $S$ -point of  $\mathbb{E}^{0|1} \rtimes \mathbb{R}^\times$ , we obtain the following.

**Lemma 2.5.** *There is a groupoid presentation,  $\mathcal{L}^{0|1}(M) \simeq \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) // \mathbb{E}^{0|1} \rtimes \mathbb{R}^\times$ .*

*Proof of Proposition 2.4.* From this groupoid presentation, the line bundle  $\omega^{1/2}$  is associated with the projection homomorphism  $\mathbb{E}^{0|1} \rtimes \mathbb{R}^\times \rightarrow \mathbb{R}^\times$  postcomposed with (31). Furthermore, sections are functions on  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  invariant under the  $\mathbb{E}^{0|1}$ -action and equivariant for the  $\mathbb{R}^\times$ -action. Under the identification,

$$C^\infty(\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)) \cong C^\infty(\text{ITM}) \cong \Omega^\bullet(M),$$

the  $\mathbb{E}^{0|1}$ -invariant functions are closed forms, and since  $\mathbb{R}^\times$  acts by degree on forms, sections of  $\omega^{\otimes k/2}$  are closed differential forms of degree  $k$ ,

$$(32) \quad \Gamma(\mathcal{L}^{0|1}(M); \omega^{\otimes k/2}) \cong \Omega_{\text{cl}}^k(M).$$

proving the proposition. □

**2.3. The Thom cocycle.** Let  $p: V \rightarrow M$  be a real vector bundle with metric  $\langle -, - \rangle$  and compatible connection  $\nabla$ . To construct a Thom cocycle, consider the diagram

$$\begin{array}{ccccc} \phi^* p^* \Pi V & \longrightarrow & p^* \Pi V & \longrightarrow & \Pi V \\ \downarrow & & \downarrow \text{\scriptsize $\mathbf{x}$} & & \downarrow \\ S \times \mathbb{R}^{0|1} & \xrightarrow{\phi} & V & \xrightarrow{p} & M \end{array}$$

gotten by pulling back  $p: V \rightarrow M$  along itself and taking the parity reversal. As usual,  $\mathbf{x}$  is the tautological section. Equip sections  $\Psi \in \Gamma(S \times \mathbb{R}^{0|1}, \phi^* p^* \Pi V)$  with the functional (16). For  $i_0: S \hookrightarrow S \times \mathbb{R}^{0|1}$  the inclusion at  $0 \in \mathbb{R}^{0|1}$ , we define component fields,

$$(33) \quad \psi_1 := i_0^* \Psi \in \Gamma(S, i_0^* \phi^* \Pi V), \quad \psi_0 := i_0^* (\nabla_D \Psi) \in \Gamma(S, i_0^* \phi^* V)$$

so that  $\Psi = \psi_1 + \theta \psi_0$ . We also obtain components for  $\phi^* \mathbf{x}$  as

$$(34) \quad \mathbf{x}_0 := i_0^* \phi^* \mathbf{x}, \quad \mathbf{x}_1 := i_0^* (\phi^* \nabla_D) \phi^* \mathbf{x} = i_0^* \iota_D \phi^* (\nabla \mathbf{x}).$$

These are vector-valued Taylor expansions in the odd variable on  $\mathbb{R}^{0|1}$ , and accordingly we write, e.g.,  $\Psi = \psi_1 + \theta \psi_0$ . This formula is one of sections  $\phi^* \Pi V$  on  $S \times \mathbb{R}^{0|1}$ , where we pull back  $\psi_1$  and  $\psi_0$  along the projection  $S \times \mathbb{R}^{0|1} \rightarrow S$ .

**Lemma 2.6.** *Pulled back to the atlas,  $\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M) \rightarrow \mathcal{L}^{0|1}(M)$  the value of  $\mathcal{S}_{\text{MQ}}$  is*

$$\mathcal{S}_{\text{MQ}}(\psi_1, \psi_0) = \frac{1}{2} (\langle \psi_0, \psi_0 - i\mathbf{x} \rangle + \langle \psi_1, F\psi_1 + i\nabla \mathbf{x} \rangle) \in C^\infty(\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, V))$$

where we have identified the endomorphism-valued curvature 2-form  $F$  with an  $\text{End}(V)$ -valued function on  $\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M)$  pulled back to  $\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, V)$ , and similarly we identify  $\nabla \mathbf{x} \in \Omega^1(V, p^* V)$  and  $\mathbf{x} \in \Omega^0(V, p^* V)$  with  $p^* V$ -valued functions on  $\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, V)$ .

*Proof.* First we compute over generic  $S$ -points, writing the component fields of  $\nabla_D \Psi$  as

$$\begin{aligned} (\nabla_D \Psi)_0 &= i_0^* \nabla_D \Psi = \psi_0, \\ (\nabla_D \Psi)_1 &= i_0^* \nabla_D (\nabla_D \Psi) = i_0^* \frac{1}{2} (\nabla_D \nabla_D + \nabla_D \nabla_D) = i_0^* \frac{1}{2} \phi^* F(D, D) \Psi \\ &= \frac{1}{2} (i_0^* \phi^* F(D, D)) \psi_1 \end{aligned}$$

where  $F$  is the curvature 2-form of  $\nabla$ . Then

$$\begin{aligned} \mathcal{S}_{\text{MQ}}(\psi_1, \psi_0) &= \frac{1}{2} \int_{S \times \mathbb{R}^{0|1}/S} (\langle \psi_1 + \theta \psi_0, \psi_0 + \theta i_0^* ((\phi^* F)(D, D)) \psi_1 \rangle - i \langle \psi_1 + \theta \psi_0, \mathbf{x}_0 + \theta \mathbf{x}_1 \rangle) d\theta \\ &= \frac{1}{2} \left( \langle \psi_0, \psi_0 - i\mathbf{x}_0 \rangle + \langle \psi_1, -\frac{1}{2} i_0^* ((\phi^* F)(D, D)) \psi_1 + i \cdot i_0^* (\phi^* (\nabla \mathbf{x})(D)) \rangle \right). \end{aligned}$$

Setting  $S = \underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, V)$ , the remaining identification is achieved by the next lemma.  $\square$

**Lemma 2.7.** *For  $S = \underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M)$  and  $\phi = \text{ev}$ , the section of the endomorphism bundle  $-\frac{1}{2} i_0^* \phi^* F(D, D)$  is the curvature 2-form,*

$$F \in \Omega^2(M; \text{End}(V)) \subset \Omega^\bullet(M; V) \cong \Gamma(\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M); \text{End}(p^* V)),$$

viewed as an  $\text{End}(V)$ -valued function on  $\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M)$ . Similarly, on  $\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, V)$ , the section-valued functions  $\mathbf{x}_1 = i_0^* (\phi^* (\nabla \mathbf{x})(D))$  and  $\mathbf{x}_0$  are identified with  $\nabla \mathbf{x} \in \Omega^1(M; V)$  and  $\mathbf{x} \in \Omega^0(M; V)$ , respectively.

*Proof.* The map  $i_0$  in this case is the inclusion  $\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M) \hookrightarrow \underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M) \times \mathbb{R}^{0|1}$  that on functions sets the coordinate  $\theta \in C^\infty(\mathbb{R}^{0|1})$  to zero. The evaluation map on functions is

$$f \mapsto f + \theta \delta f \in C^\infty(\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M) \times \mathbb{R}^{0|1}), \quad f \in C^\infty(M)$$

where  $\delta f \in C^\infty(\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M))$  is the function associated to the 1-form  $df \in \Omega^1(M)$ ; we use different notation as we will need to distinguish from the de Rham operator  $d$  on  $\Omega^\bullet(\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M))$  momentarily.

For a 2-form  $F = fdgdh \in \Omega^2(M)$ , pull back along evaluation is

$$\text{ev}^*F = \text{ev}^*(fdgdh) = (f + \theta\delta f)d(f + \theta\delta f)d(f + \theta\delta g),$$

and so

$$i_0^*((\phi^*F)(D, D)) = i_0^*(\iota_D \iota_D \text{ev}^*F) = -2f\delta g\delta h = -2F.$$

From this (and, e.g., working in a local trivialization of  $V$ ), we yield the claimed formula. The arguments yielding  $\nabla \mathbf{x}$  and  $\mathbf{x}$  are similar.  $\square$

*Proof of Proposition 1.7.* It remains to evaluate the integral over sections. We have

$$\begin{aligned} \int e^{-S_{\text{MQ}}(\Psi)} d\Psi &= \int e^{-S_{\text{MQ}}(\psi_1, \psi_0)} d\psi_1 d\psi_0 = \int e^{-\frac{1}{2}\langle \psi_0, \psi_0 - i\mathbf{x} \rangle} d\psi_0 \int e^{-\frac{1}{2}\langle \psi_1, F\psi_1 + i\nabla \mathbf{x} \rangle} d\psi_1 \\ &= (2\pi)^{\dim(V)/2} e^{-\|\mathbf{x}\|^2/2} \int e^{-\frac{1}{2}\langle \psi_1, F\psi_1 + i\nabla \mathbf{x} \rangle} d\psi_1 \end{aligned}$$

This remaining integral over the odd parameter space is precisely the Mathai–Quillen Thom form as defined in [BGV92] §1.6, up to the stated factor of  $(2\pi)^{-\dim(V)}$ .  $\square$

**2.4. Integration on  $\mathcal{L}^{0|1}(M)$  and pushforwards in de Rham cohomology.** Below we reformulate classical facts about differential forms into the super geometry of the stack  $\mathcal{L}^{0|1}(M)$ . Although there isn't much mathematical content, it illustrates how to translate the super geometry back into differential forms which will become more elaborate in subsequent sections.

For  $M$  oriented, there is a canonical volume form on  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  associated to integration of forms,

$$C^\infty(\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)) \xrightarrow{\int_M} C^\infty(\text{pt}) \cong \mathbb{C}, \quad f \in C^\infty(\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)) \cong \Omega^\bullet(M).$$

More generally, for a smooth fiber bundle  $\pi: M \rightarrow B$  with oriented fibers, we get a volume form on fibers of  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \rightarrow \underline{\text{SMfld}}(\mathbb{R}^{0|1}, B)$ . Stokes' theorem implies that these fiberwise volume forms descend to volume forms on the fibers of the map of stacks

$$\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) // \mathbb{E}^{0|1} \rightarrow \underline{\text{SMfld}}(\mathbb{R}^{0|1}, B) // \mathbb{E}^{0|1}.$$

Said plainly: integrals of closed forms are closed. However, these volume forms don't generally descend to the stack  $\mathcal{L}^{0|1}(M) \simeq \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) // \underline{\text{Iso}}(\mathbb{R}^{0|1})$ , coming from the fact that integration shifts degrees of differential forms. By inspection, integration determines a map on sections of line bundles,

$$\Gamma(\mathcal{L}^{0|1}(X), \omega^{\bullet/2}) \rightarrow \Gamma(\mathcal{L}^{0|1}(M), \omega^{(\bullet-n)/2})$$

where  $n$  is the fiber dimension of  $\pi$ . We call this map the *analytic pushforward*, denoted  $\pi_!^{\text{an}}$ , to distinguish it from the pushforward defined below using the Thom isomorphism.

For vector bundles  $V \rightarrow M$  with metric and compatible connection we get a section,

$$\text{Th}(V) \in \Gamma_{\text{cvs}}(\mathcal{L}^{0|1}(V), \omega^{\dim(V)})$$

such that the map

$$\Gamma(\mathcal{L}^{0|1}(M), \omega^{\bullet/2}) \xrightarrow{\otimes \text{Th}(V)} \Gamma_{\text{cvs}}(\mathcal{L}^{0|1}(V), \omega^{(\bullet+\dim(V))/2})$$

induces an isomorphism on concordance classes. This is a rephrasing of the Thom isomorphism. For a family of oriented manifolds,  $\pi: M \rightarrow B$ , an embedding  $M \hookrightarrow \mathbb{R}^N$  determines an embedding  $i: M \hookrightarrow B \times \mathbb{R}^N$ . A choice of tubular neighborhood  $\nu$  for  $i$  gives maps

$$\begin{aligned} \Gamma(\mathcal{L}^{0|1}(M), \omega^{\bullet/2}) &\xrightarrow{\otimes \text{Th}(\nu)} \Gamma_{\text{cs}}(\mathcal{L}^{0|1}(\nu); \omega^{(\bullet+N-n)/2}) \rightarrow \Gamma_{\text{cs}}(\mathcal{L}^{0|1}(B \times \mathbb{R}^N); \omega^{(\bullet+N-n)/2}) \\ &\xrightarrow{\int_{\mathbb{R}^N}} \Gamma_{\text{cs}}(\mathcal{L}^{0|1}(B); \omega^{(\bullet-n)/2}) \end{aligned}$$

where the first map is the cocycle-level Thom isomorphism, the second map extends functions by zero, and the third map integrates over  $\mathbb{R}^N$ . We call this composition the *topological pushforward* and denote it by  $\pi_!^{\text{top}}$ .

The maps  $\pi_1^{\text{an}}$  and  $\pi_1^{\text{top}}$  are in fact equal, which in this case boils down to the fact that the integral of the Thom cocycle over the fibers of a vector bundle is 1. Our reason for distinguishing them is that analogous pushforwards associated to 1|1- and 2|1-dimensional field theories are not obviously equal, as the analytic constructions and physical intuition leading to these pushforwards are quite different.

### 3. COCYCLE PUSHFORWARDS IN COMPLEXIFIED K-THEORY

In this section we prove the K-theory analog of Theorems 1.3, 1.4 and 1.5. We review the super loop space differential cocycle model for  $K \otimes \mathbb{C}$  in §3.1. In §3.3 we use this to construct the K-theoretic Mathai–Quillen form, which in turn defines the topological pushforward. Next, in §3.4, we construct a cocycle representative for the  $\hat{A}$ -class of a geometric family  $\pi: M \rightarrow B$ , defining the analytic pushforward. This is a straightforward generalization of Alvarez-Gaumé’s [AG83] construction of the  $\hat{A}$ -class of an oriented manifold (i.e., the case  $M = \text{pt}$ ) that defines the local index in the physical proof of the index theorem. We conclude this section by formulating the equality of the topological and analytic pushforwards in  $K \otimes \mathbb{C}$  in terms of an equality of  $\zeta$ -super determinants for families of operators on the moduli stack  $\mathcal{L}_0^{1|1}(X)$  of constant super loops.

**3.1. Super loop spaces and complexified K-theory.** The cocycle model for  $K \otimes \mathbb{C}$  below is a mild reworking of the one in [BE13, §2]. For completeness we include brief proofs.

**Definition 3.1.** The 1|1-dimensional rigid conformal model geometry takes  $\mathbb{R}^{1|1}$  as its model space and the super group  $\mathbb{E}^{1|1} \rtimes \mathbb{R}^\times$  as isometry group, where  $\mathbb{E}^{1|1}$  is  $\mathbb{R}^{1|1}$  as a supermanifold with multiplication

$$(t, \theta) \cdot (t', \theta') = (t + t' + i\theta\theta', \theta + \theta'), \quad (t, \theta), (t', \theta') \in \mathbb{R}^{1|1}(S),$$

and the semidirect product  $\mathbb{E}^{1|1} \rtimes \mathbb{R}^\times$  comes from the action  $\mu \cdot (t, \theta) = (\mu^2 t, \mu\theta)$ , for  $(t, \theta) \in \mathbb{E}^{1|1}(S)$  and  $\mu \in \mathbb{R}^\times(S)$ . We take the obvious left action of  $\mathbb{E}^{1|1} \rtimes \mathbb{R}^\times$  on  $\mathbb{R}^{1|1}$ . The Lie algebra of left-invariant vector fields on  $\mathbb{E}^{1|1}$  has a single odd generator,  $D := \partial_\theta - i\theta\partial_t$  that squares to  $-i\partial_t$ .

*Remark 3.2.* The 1|1-dimensional conformal model geometry has as isometries all diffeomorphisms of  $\mathbb{R}^{1|1}$  that preserve the distribution generated by  $D$  [Fre99]. Hence, the rigid conformal isometry group is a strict subgroup of the conformal isometry group.

A family of 1-dimensional (oriented) lattices is a homomorphism over  $S$ ,  $\langle r \rangle: S \times \mathbb{Z} \rightarrow S \times \mathbb{E}$  so that the image  $S \times \{1\} \subset S \times \mathbb{Z} \rightarrow S \times \mathbb{R}$  is determined by an  $S$ -point  $r \in \mathbb{R}_{>0}(S) \subset \mathbb{R}(S)$ . Through the inclusion of groups  $\mathbb{E} \subset \mathbb{E}^{1|1}$ , an  $S$ -family of lattices defines a family of *super circles* via the quotient  $S \times \mathbb{R}^{1|1} / \langle r \rangle =: S \times_r \mathbb{R}^{1|1}$ .

**Definition 3.3.** The *super loop stack* of  $M$ , denoted  $\mathcal{L}^{1|1}(M)$ , has as objects over  $S$  pairs  $(r, \phi)$  where  $r \in \mathbb{R}_{>0}(S)$  determines a family of super circles  $S \times_r \mathbb{R}^{1|1}$  and  $\phi: S \times_r \mathbb{R}^{1|1} \rightarrow M$  is a map. Morphisms between these objects over  $S$  consist of commuting triangles

$$(35) \quad \begin{array}{ccc} S \times_r \mathbb{R}^{1|1} & \xrightarrow{\cong} & S \times_{r'} \mathbb{R}^{1|1} \\ \searrow \phi & & \swarrow \phi' \\ & M & \end{array}$$

where the horizontal arrow is an isomorphism of  $S$ -families of rigid conformal super manifolds. The stack of *constant super loops*, denoted  $\mathcal{L}_0^{1|1}(M)$ , is the full substack for which  $\phi$  is invariant under loop rotation, i.e.,  $(r, \phi)$  is an  $S$ -point of  $\mathcal{L}_0^{1|1}(M)$  if for families of isometries associated with sections of the bundle of groups  $S \times_r \mathbb{E} \rightarrow S$ , the triangle (35) commutes with  $r = r'$  and  $\phi = \phi'$ .

Define a morphism of stacks  $\mathcal{L}^{1|1}(\text{pt}) \rightarrow \text{pt}/\mathbb{Z}/2$  that is constant to  $\text{pt}$  on objects over  $S$  and for an isometry  $S \times_r \mathbb{R}^{1|1} \rightarrow S \times_r \mathbb{R}^{1|1}$  encodes whether the orientations on  $S \times \mathbb{R}^{0|1} \subset S \times_r \mathbb{R}^{1|1}$  are preserved or reversed, i.e., for the map  $S \rightarrow \mathbb{R}^\times$  associated to the rigid conformal isometry, we postcompose with the sign map,  $\mathbb{R}^\times \rightarrow \{\pm 1\} \cong \mathbb{Z}/2$ . Define an odd complex line bundle over  $\text{pt}/\mathbb{Z}/2$  from the inclusion

$$\mathbb{Z}/2 \cong \{\pm 1\} \subset \mathbb{C}^\times \cong \text{GL}(\mathbb{C}^{0|1}).$$

**Definition 3.4.** Define a line bundle  $\omega^{m/2}$  on  $\mathcal{L}_0^{1|1}(M)$  as the pullback of  $m$ th tensor power of the odd line bundle over  $\text{pt}/\mathbb{Z}/2$  along the composition,

$$\mathcal{L}_0^{1|1}(M) \rightarrow \mathcal{L}_0^{1|1}(\text{pt}) \cong \mathcal{L}^{1|1}(\text{pt}) \rightarrow \text{pt}/\mathbb{Z}/2.$$

### 3.2. An atlas for $\mathcal{L}^{1|1}(M)$ and its groupoid presentation.

**Lemma 3.5.** *A map  $\phi$  is invariant under loop rotation if and only if it factors through the map  $S \times_r \mathbb{R}^{1|1} \rightarrow S \times \mathbb{R}^{0|1}$  induced by the projection  $\mathbb{R}^{1|1} \rightarrow \mathbb{R}^{0|1}$ .*

*Proof.* A map  $\phi$  being invariant under the action is equivalent to it factoring through the fiberwise quotient. But this is the same as the quotient by the  $\mathbb{E}^1$ -action, and this is easily computed as

$$(S \times_r \mathbb{R}^{1|1})/\mathbb{E}^1 \cong S \times \mathbb{R}^{0|1},$$

which identifies invariance under loop rotation with the factorization property.  $\square$

For an  $S$ -point of the constant super loops, this factorization property means  $\phi$  is determined by a map  $\phi_0: S \times \mathbb{R}^{0|1} \rightarrow M$ . This gives an atlas

$$(36) \quad u: \mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \rightarrow \mathcal{L}_0^{1|1}(M),$$

whose associated groupoid presentation we describe presently. Since super circles are a quotient of  $\mathbb{R}^{1|1}$  there is an exact sequence

$$(37) \quad r\mathbb{Z} \hookrightarrow \underline{\text{Iso}}(\mathbb{R}^{1|1}) \twoheadrightarrow \underline{\text{Iso}}(\mathbb{R}^{1|1}/r\mathbb{Z}).$$

The action by translations  $\mathbb{E}^{1|1} < \underline{\text{Iso}}(\mathbb{R}^{1|1})$  leaves the lattice  $r\mathbb{Z} \subset \mathbb{R}^{1|1}$  unchanged, whereas the target lattice of a dilation  $\mu \in \mathbb{R}^\times$  is  $\mu^2 r$ . This proves the following.

**Proposition 3.6.** *There is an essentially surjective full morphism of stacks,*

$$\mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)/\underline{\text{Iso}}(\mathbb{R}^{1|1}) \rightarrow \mathcal{L}_0^{1|1}(M),$$

where the  $\underline{\text{Iso}}(\mathbb{R}^{1|1})$ -action on  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  is through the homomorphism  $\underline{\text{Iso}}(\mathbb{R}^{1|1}) \twoheadrightarrow \mathbb{E}^{0|1} \rtimes \mathbb{R}^\times$  followed by the precomposition action on  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$ , and the action on  $\mathbb{R}_{>0}$  is through the projection to  $\mathbb{R}^\times$  followed by the dilation action on  $\mathbb{R}_{>0}$ . This induces an equivalence of stacks,

$$\left( \begin{array}{c} (\underline{\text{Iso}}(\mathbb{R}^{1|1}) \times \mathbb{R}_{>0})/\mathbb{Z} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \\ \Downarrow \\ \mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \end{array} \right) \xrightarrow{\sim} \mathcal{L}_0^{1|1}(M),$$

where the quotient by  $\mathbb{Z}$  is the quotient by the fiberwise kernel of (37).

The homomorphism

$$\underline{\text{Iso}}(\mathbb{R}^{1|1}) \cong \mathbb{E}^{1|1} \rtimes \mathbb{R}^\times \twoheadrightarrow \mathbb{R}^\times \xrightarrow{\text{sgn}} \{\pm 1\} \subset \mathbb{C}^\times \cong \text{GL}(\mathbb{C}^{0|1})$$

determines a line bundle over the Lie groupoid in the statement of (3.6) that is isomorphic to the pullback of  $\omega^{1/2}$  from  $\mathcal{L}_0^{1|1}(M)$ . This allows us to compute sections in terms of functions on the atlas with transformation properties.

**Proposition 3.7.** *Sections of  $\omega^{n/2}$  are spanned by functions on  $\mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  of the form*

$$(38) \quad r^{k/2} \otimes f \in C^\infty(\mathbb{R}_{>0}) \otimes \Omega_{\text{cl}}^k(M) \subset C^\infty(\mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)).$$



where  $r$  is a coordinate on  $\mathbb{R}_{>0}$ , and  $n$  and  $k$  are required to have the same parity. In particular, we get isomorphisms

$$\Gamma(\mathcal{L}_0^{1|1}(M); \omega^{\bullet/2}) \cong \begin{cases} \Omega_{\text{cl}}^{\text{ev}}(M) & \bullet \text{ even} \\ \Omega_{\text{cl}}^{\text{odd}}(M) & \bullet \text{ odd} \end{cases}$$

compatible with the multiplications.

*Proof.* The isomorphism of groups  $\mathbb{R}^\times \cong \mathbb{R}_{>0} \times \{\pm 1\} \cong \mathbb{R}_{>0} \times \mathbb{Z}/2$  allows us to compute sections of  $\omega^{n/2}$  as functions on  $\mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  that are invariant under the  $\mathbb{E}^{1|1} \rtimes \mathbb{R}_{>0}$ -action and equivariant for the  $\mathbb{Z}/2$ -action. The  $\mathbb{E}^{1|1}$ -action is generated by the de Rham operator on  $C^\infty(\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)) \cong \Omega^\bullet(M)$ . The  $\mathbb{R}_{>0}$ -action is diagonal on  $\mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$ , with invariants being generated by functions of the form  $r^{k/2} \otimes f$  as claimed in the proposition. Finally, equivariance for the  $\mathbb{Z}/2$ -action specifies that the degree of  $f \in \Omega_{\text{cl}}^k(M)$  have the same parity as  $n$ .  $\square$

**3.3. The Mathai–Quillen form.** To construct a Thom cocycle, consider the diagram

$$(39) \quad \begin{array}{ccccc} \phi^* p^* \Pi V & \longrightarrow & p^* \Pi V & \longrightarrow & \Pi V \\ \downarrow & & \downarrow \mathbf{x} & & \downarrow \\ S \times_r \mathbb{R}^{1|1} & \xrightarrow{\phi} & V & \xrightarrow{p} & M. \end{array}$$

Define a vector bundle  $\mathcal{F}^{1|1}(V)$  whose fiber at an  $S$ -point  $(r, \phi)$  is the  $C^\infty(S)$ -module  $\Gamma(S \times_r \mathbb{R}^{1|1}, \phi^* p^* V)$ . Pulling this module back along isometries of super circles defines a vector bundle over the stack  $\mathcal{L}_0^{1|1}(M)$ . The metric and connection on  $V$  pull back to these spaces of sections. There is a function on sections given by essentially the same formula as in the  $0|1$ -dimensional case

$$\mathcal{S}_{\text{MQ}}(\Psi) = \frac{1}{2} \int_{S \times_r \mathbb{R}^{1|1}/S} \left( \langle \Psi, \nabla_D \Psi \rangle - \frac{i}{\sqrt{r}} \langle \Psi, \phi^* \mathbf{x} \rangle \right) d\theta dt$$

except now  $D = \partial_\theta - i\partial_t$  and we rescale the second term by the volume of the super circle in question. This rescaling guarantees that  $\mathcal{S}_{\text{MQ}}$  is invariant under dilations of super circles, so really is a function on the stack. We define component fields of  $\Psi$  and  $\phi^* \mathbf{x}$  as before,

$$(40) \quad \begin{aligned} (\nabla_D \Psi)_0 &= i_0^* \nabla_D \Psi = \psi_0 \\ (\nabla_D \Psi)_1 &= i_0^* (\nabla_D \nabla_D \Psi) = i_0^* (\tfrac{1}{2} \phi^* F(D, D) - \nabla_{D^2}) \Psi = (\tfrac{1}{2} i_0^* \phi^* F + i \nabla_{\partial_t}) \psi_1. \\ (\phi^* \mathbf{x})_0 &= i_0^* \phi^* \mathbf{x} \\ (\phi^* \mathbf{x})_1 &= i_0^* \phi^* \nabla_D \phi^* \mathbf{x} = i_0^* (\iota_D \phi^* (\nabla \mathbf{x})). \end{aligned}$$

Equations like  $\Psi = \psi_1 + \theta \psi_0$  are equalities between sections of  $\phi^* \Pi V$  on  $S \times_r \mathbb{R}^{1|1}$ , where we pull back  $\psi_1$  and  $\psi_0$  along the projection  $S \times_r \mathbb{R}^{1|1} \rightarrow S \times_r \mathbb{R}$ .

**Proposition 3.8.** *Evaluated on the universal family  $S = \mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, V)$ , the functional  $\mathcal{S}_{\text{MQ}}$  takes the form*

$$\mathcal{S}_{\text{MQ}}(\psi_1, \psi_0) = \frac{1}{2} \int_{S \times_r \mathbb{R}} \left( \langle \psi_0, \psi_0 - \frac{i}{\sqrt{r}} \mathbf{x} \rangle + \langle \psi_1, (-i \nabla_{\partial_t} + F) \psi_1 + \frac{i}{\sqrt{r}} \nabla \mathbf{x} \rangle \right) dt$$

where we again have abused notation to regard  $\mathbf{x}, \nabla \mathbf{x}$  and  $F$  as section- or endomorphism-valued functions on  $\mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, V)$ .

*Proof.* This follows immediately from computing the Berezinian integral with respect to  $\theta$  and applying Lemma 2.7.  $\square$

Now we set to work on providing rigorous meaning for the integral  $\int e^{-\mathcal{S}_{\text{MQ}}(\Psi)} d\Psi$  over sections, as per (21). We pullback  $\mathcal{F}^{1|1}(V)$  along (36) and split it into a finite-dimensional piece,  $\mathcal{F}_0^{1|1}(V)$ , and an infinite-dimensional one,  $\mathcal{F}_\perp^{1|1}(V)$ . This splitting is orthogonal with

respect to a pairing described in component fields using the fiberwise volume form on  $S \times_r \mathbb{R}$  as

$$(41) \quad \int_{S \times_r \mathbb{R}/S} \langle \psi_0, \psi'_0 \rangle dt, \quad \int_{S \times_r \mathbb{R}/S} \langle \psi_1, \psi'_1 \rangle dt,$$

i.e., we take the usual  $L^2$ -inner product for sections over  $S^1$ .

**Definition 3.9.** Define the bundle of *zero modes*  $\mathcal{F}_0^{1|1}(V) \subset u^* \mathcal{F}^{1|1}(V)$  as the subbundle whose sections satisfy  $\nabla_{\partial_t} \Psi = 0$ . Let  $\mathcal{F}_\perp^{1|1}(V)$  denote the bundle whose value at an  $S$ -point is the orthogonal complement to  $\mathcal{F}_0^{1|1}(V)$  in  $u^* \mathcal{F}^{1|1}(V)$  using the metric on component fields defined above.

We now show that the (finite-dimensional) integral of  $\mathcal{S}_{\text{MQ}}$  over  $\mathcal{F}_0^{1|1}(V)$  agrees with the Mathai–Quillen form in de Rham cohomology.

*Proof of Lemma 1.8 when  $d = 1$ .* For sections  $\Psi = \psi_1 + \theta \psi_0$  of  $\mathcal{F}_0^{1|1}(V)$  at an  $S$ -point  $(r, \phi)$ ,

$$\begin{aligned} \mathcal{S}_{\text{MQ}}(\psi_1, \psi_0) &= \int_{S \times_r \mathbb{R}} \left( \langle \psi_0, \psi_0 - \frac{i}{\sqrt{r}} \mathbf{x} \rangle + \langle \psi_1, (-i \nabla_{\partial_t} + F) \psi_1 + \frac{i}{\sqrt{r}} \nabla \mathbf{x} \rangle \right) dt \\ &= r \left( \langle \psi_0, \psi_0 - \frac{i}{\sqrt{r}} \mathbf{x} \rangle + \langle \psi_1, F \psi_1 + \frac{i}{\sqrt{r}} \nabla \mathbf{x} \rangle \right) \in C^\infty(S) \end{aligned}$$

where we used the integrand is constant on the fibers of  $S \times_r \mathbb{R}$  over  $S$ , so the integral is simply the fiberwise volume multiplied by the integrand. The (finite-dimensional) Gaussian integral over zero modes is then the product

$$\int_{\mathcal{F}_0^{1|1}(V)} e^{-\mathcal{S}_{\text{MQ}}(\Psi)} d\Psi = \int \exp(-r \langle \psi_0, \psi_0 - \frac{i}{\sqrt{r}} \mathbf{x} \rangle) d\psi_0 \int \exp(-r \langle \psi_1, F \psi_1 + \frac{i}{\sqrt{r}} \nabla \mathbf{x} \rangle) d\psi_1.$$

The first factor is a standard Gaussian, and evaluates as

$$\int \exp(-r \langle \psi_0, \psi_0 - \frac{i}{\sqrt{r}} \mathbf{x} \rangle) d\psi_0 = \left( \frac{2\pi}{r} \right)^{\dim(V)/2} \exp(-\|x\|^2/2).$$

Writing the integral over  $\psi_1$  as

$$\int \exp(-r \langle \psi_1, F \psi_1 - \frac{i}{\sqrt{r}} \nabla \mathbf{x} \rangle) d\psi_1 = \int \exp(-\langle \psi_1, r \cdot F \psi_1 - i \sqrt{r} \nabla \mathbf{x} \rangle) d\psi_1,$$

we identify it with the Mathai–Quillen form (4) in de Rham cohomology under the inclusion of closed forms into  $C_{\text{cs}}^\infty(\mathcal{L}_0^{1|1}(V))$  determined by (38).  $\square$

Next we analyze  $\mathcal{S}_{\text{MQ}}$  on  $\mathcal{F}_\perp^{1|1}(V)$ , verifying (18) and (19).

**Lemma 3.10.** *For sections  $\Psi$  of  $\mathcal{F}_\perp^{1|1}(V)$ , we have*

$$(42) \quad \mathcal{S}_{\text{MQ}}(\Psi) = \int_{S \times_r \mathbb{R}^{1|1}} \langle \Psi, \mathcal{D}_V^{1|1} \Psi \rangle,$$

where  $\mathcal{D}_V^{1|1}$  is the restriction of  $\nabla_D$ . In components,

$$(43) \quad \mathcal{S}_{\text{MQ}}(\psi_1, \psi_0) = \int_{S \times_r \mathbb{R}} (\langle \psi_0, \psi_0 \rangle + \langle \psi_1, (-i \nabla_{\partial_t} + F) \psi_1 \rangle) dt$$

for  $-i \nabla_{\partial_t} + F$  a family of invertible operators.

*Proof.* The bundle  $\mathcal{F}^{1|1}(V)$  at an  $S$ -point  $(r, \phi)$  is pulled back along  $S \times_r \mathbb{R}^{1|1} \rightarrow S \times \mathbb{R}^{0|1} \rightarrow V$ , so sections are spanned by functions on the fiber,  $S \times_r \mathbb{R}$  tensored with sections of  $i_0^* \phi^* V$  pulled back to  $S \times \mathbb{R}^{0|1}$ . Similarly, sections of  $\mathcal{F}_\perp^{1|1}(V)$  are functions on  $S \times_r \mathbb{R}$  with nonzero Fourier modes tensored with sections of  $i_0^* \phi^* V$  pulled back to  $S \times \mathbb{R}^{0|1}$ . This implies that

$$\int_{S \times_r \mathbb{R}} \langle \psi_0, (\phi^* \mathbf{x})_0 \rangle = 0, \quad \int_{S \times_r \mathbb{R}} \langle \psi_1, (\phi^* \mathbf{x})_1 \rangle = 0$$

for  $\Psi = \psi_1 + \theta\psi_0$  a section of  $\mathcal{F}_\perp^{1|1}(V)$  at the  $S$ -point  $(r, \phi)$ . Hence, the action reduces to the claimed form (42). Since  $-i\nabla_{\partial_t}$  is an invertible operator on  $\mathcal{F}_\perp^{1|1}(V)$ , and  $-i\nabla_{\partial_t} + F$  is a nilpotent modification of this invertible operator,  $-i\nabla_{\partial_t} + F$  is invertible.  $\square$

*Proof of 1.9.* By definition, the  $\zeta$ -super determinant is a ratio of  $\zeta$ -regularized determinants applied to operators acting on even and odd sections of  $\mathcal{F}_\perp^{1|1}(V)$  pulled back to the atlas  $\mathbb{R}_{>0} \times \underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, V)$ . The relevant operator on the even sections  $\psi_0$  is the identity, so this contributes 1 to the  $\zeta$ -regularized super determinant.

On odd sections  $\psi_1$ , choose a basis  $e^{2\pi i n t/r} \otimes v$  for  $v$  a section of  $i_0^* \text{ev}^* \text{IV}$ . In this basis, the operator  $\nabla_{\partial_t}$  is just  $\partial_t \otimes 1$ , as  $\nabla$  is pulled back along a constant map. Restricted to the subspace spanned by sections of this form for a fixed  $n$ ,  $-i\nabla_{\partial_t} + F$  acts by  $2\pi n/r + F$ . Hence, we have the  $\zeta$ -function

$$\zeta_{\mathcal{D}}(s) = \sum_n \text{Tr} (2\pi n t/r + F)^s$$

Binomial expansion gives

$$\begin{aligned} \zeta_{\mathcal{D}}(s) &= \sum_{n \neq 0} \text{Tr} \left( \text{id} - F \otimes \frac{r}{2\pi n} \right)^s \left( \frac{2\pi n}{r} \right)^s \\ &= \sum_{n \neq 0} \sum_{k=0}^{\text{finite}} \text{Tr} \left( F^k \frac{s(s-1) \cdots (s-k+1)}{k! (2\pi n)^k} r^k \right) \left( \frac{2\pi n}{r} \right)^s \end{aligned}$$

where the sum over  $k$  is finite because  $F$  is nilpotent. For  $k > 1$ , we differentiate under the sum and obtain the contribution to  $\zeta'_{\mathcal{D}}(0)$

$$\sum_{k=1}^{\text{finite}} \text{Tr} (F^k) \frac{(-1)^{k-1}}{k(2\pi)^k} r^k 2\zeta(k) = - \sum_{k=1}^{\infty} \frac{\text{Tr}(F^{2k}) r^{2k}}{2k(2\pi)^{2k}} 2\zeta(2k)$$

where  $\zeta(k)$  denotes the value of the Riemann  $\zeta$ -function at  $k$ , and we have used that traces of odd powers of  $F$  vanish. The  $k = 0$  part is a standard  $\zeta$ -regularized product, and equal to  $(\frac{2\pi}{r})^{n/2}$ , e.g., see Example 2 of [QHS93]. So together we have

$$\text{sdet}_{\zeta}(\mathcal{D}_V^{d|1}) = \left( \frac{2\pi}{r} \right)^{n/2} \exp \left( - \sum_{k=1}^{\infty} \frac{\text{Tr}(F^{2k}) r^{2k}}{2k(2\pi)^{2k}} \zeta(2k) \right).$$

In our cochain model,

$$(44) \quad r^{2k} \text{Tr}(F^{2k}) = 2(2k)! \text{ph}_k(V),$$

where  $\text{ph}_k$  denotes the  $4k^{\text{th}}$  component of the Pontryagin character as a function on  $\mathcal{L}_0^{1|1}(M)$ . Putting this together we get

$$\text{sdet}_{\zeta}(\mathcal{D}_V^{d|1}) = \left( \frac{2\pi}{r} \right)^{n/2} \exp \left( - \sum_{k=1}^{\infty} \frac{(2k)! \text{ph}_k(V)}{2k} \frac{2\zeta(2k)}{(2\pi)^{2k}} \right)$$

which we identify with  $(2\pi/r)^{n/2}$  multiplied by  $\hat{A}(V)^{-1}$  as an element of  $C^\infty(\mathcal{L}_0^{1|1}(M))$  pulled back to  $C^\infty(\mathcal{L}_0^{1|1}(V))$ .

This  $\zeta$ -Berezinian combined with Lemma 1.8 (for  $d = 1$ ) shows that

$$\int_{\mathcal{F}_0^{d|1}(V)} \exp(-\mathcal{S}_{\text{MQ}}(\Psi_0)) d\Psi_0 \cdot \frac{\text{sdet}_{\zeta}(\mathcal{D}_V^{1|1})}{Z^n} \in \Gamma_{\text{cs}}(\mathcal{L}_0^{1|1}(V); \omega^{\dim(V)/2})$$

is a cocycle representative for the K-theoretic Thom class, where  $Z = (2\pi/r)^{1/2}$ . Note that this is both the unique normalization making the above expression a section of the claimed sort, and it satisfies  $\text{sdet}_{\zeta}(\mathcal{D}_{\underline{\mathbb{R}}}^{1|1})/Z = 1$  for the trivial bundle.  $\square$

**3.4. The  $\hat{A}$ -cocycle of a geometric family  $\pi: M \rightarrow B$ .** To define the operators  $\Delta_{M/B}^{1|1}$ , first we specify the vector bundle on which they act. This essentially comes from a notion of tubular neighborhood for the inclusion  $\mathcal{L}_0^{1|1}(M) \subset \mathcal{L}^{1|1}(M)$  for which we restrict to normal directions along the fibers of the map  $\mathcal{L}^{1|1}(M) \rightarrow \mathcal{L}^{1|1}(B)$  induced by  $\pi$ ; see §1.7.

For  $M \rightarrow B$  a geometric family of oriented manifolds, define an infinite-rank vector bundle  $\mathcal{T}^{1|1}(M/B) \rightarrow \mathcal{L}^{1|1}(M)$  whose fiber at an  $S$ -point is the  $C^\infty(S)$ -module  $\Gamma(S \times_r \mathbb{R}^{1|1}, \phi^*T(M/B))$ . The metric and connection on  $T(M/B)$  (which comes from the data of a geometric family) pull back to these spaces of sections. We define component fields

$$(45) \quad a := i_0^* \sigma, \quad \eta = i_0^*(\phi^* \nabla)_D \sigma$$

for  $\sigma$  a section,  $D = \partial_\theta - i\theta \partial_t$  and  $i_0: S \times_r \mathbb{R} \hookrightarrow S \times_r \mathbb{R}^{1|1}$  the inclusion of the fiberwise reduced manifold. The fiberwise volume form on  $S \times_r \mathbb{R}$  gives a pairing on sections of  $\mathcal{T}^{1|1}(M/B)$  at each  $S$ -point that in components is

$$(46) \quad \int_{S \times_r \mathbb{R}/S} \langle a, a' \rangle dt, \quad \int_{S \times_r \mathbb{R}/S} \langle \eta, \eta' \rangle dt.$$

**Definition 3.11.** Define the vector bundle  $\mathcal{N}^{1|1}(M/B) \subset \mathcal{T}^{1|1}(M/B)|_{\mathcal{L}_0^{1|1}(M)}$  over  $\mathcal{L}_0^{1|1}(M)$  as having  $S$ -points sections in the orthogonal complement of the constant sections with respect to the pairings (46), where a section  $\sigma$  is constant if  $\nabla_{\partial_t} \sigma = 0$ . We use the notation  $\Gamma_0(S \times_r \mathbb{R}^{1|1}, \phi^*T(M/B)) \subset \Gamma(S \times_r \mathbb{R}^{1|1}, \phi^*T(M/B))$  to denote this orthogonal complement at an  $S$ -point  $(r, \phi)$ .

Now we define the operators  $\Delta_{M/B}^{1|1}$  acting on sections of  $\mathcal{N}^{1|1}(M/B)$ . These come from the Hessian of the classical action for the 1|1-dimensional sigma model with target  $M$  (i.e.,  $N = 1$  supersymmetric mechanics). Define a function on sections

$$(47) \quad \text{Hess}_\phi(\sigma) := -i \int_{S \times_r \mathbb{R}^{1|1}/S} \langle \sigma, \nabla_{\partial_t} \nabla_D \sigma \rangle dt d\theta, \quad \sigma \in \Gamma_0(S \times_r \mathbb{R}^{1|1}, \phi^*T(M/B))$$

where the integral is the Berezinian integral over the fibers of the projection  $S \times_r \mathbb{R}^{1|1} \rightarrow S$ . Since it is built out of right-invariant vector fields, the function Hess is automatically invariant under the left-action of isometries. Therefore Hess defines a function on the stack as claimed. We use the notation  $\Delta_{M/B}^{1|1} = \nabla_{\partial_t} \nabla_D$  to emphasize the dependence of this family of operators on  $\pi: M \rightarrow B$ . A component-form version will facilitate computations.

**Lemma 3.12.** *Taylor expanding  $\sigma$  using (45) and performing the Berezin integral in (23),*

$$\begin{aligned} \text{Hess}_\phi(\sigma) &= \text{Hess}_\phi(a, \eta) = - \int_{S \times_r \mathbb{R}/S} \langle (\Delta_{M/B}^{1|1})^{\text{ev}} a, a \rangle + \langle (\Delta_{M/B}^{1|1})^{\text{odd}} \eta, \eta \rangle dt, \\ (\Delta_{M/B}^{1|1})^{\text{ev}} &:= -\nabla_{\partial_t}^2 + \frac{i}{2} \cdot R \nabla_{\partial_t}, \quad (\Delta_{M/B}^{1|1})^{\text{odd}} = i \cdot \nabla_{\partial_t} \end{aligned}$$

where  $R = \phi^*R(D, D)$  is the  $\text{End}(\phi^*T(M/B))$ -valued function on  $S \times_r \mathbb{R}^{1|1}$  determined by the curvature 2-form of the connection on  $T(M/B)$ .

*Proof.* We compute the Taylor components as in (45) of the section  $\nabla_{\partial_t} \nabla_D \sigma$ :

$$\begin{aligned} i_0^*(\nabla_{\partial_t} \nabla_D \sigma) &= \nabla_{\partial_t} i_0^*(\nabla_D \sigma) = \nabla_{\partial_t} \eta \\ i_0^* \nabla_D (\nabla_{\partial_t} \nabla_D \sigma) &= i_0^*(\nabla_{\partial_t} \nabla_D \nabla_D \sigma) = \frac{1}{2} i_0^*(\nabla_{\partial_t} (R(D, D) - \nabla_{[D, D]}) \sigma) \\ &= (R(D, D) \nabla_{\partial_t} + i \nabla_{\partial_t}^2) i_0^* \sigma = (i \nabla_{\partial_t}^2 + R(D, D) \nabla_{\partial_t}) a \end{aligned}$$

where we used

$$\nabla_D^2 = \frac{1}{2} (\nabla_D \nabla_D + \nabla_D \nabla_D) = \frac{1}{2} (R(D, D) - \nabla_{[D, D]}) = \frac{1}{2} R(D, D) + i \nabla_{\partial_t}.$$

So now we have

$$\begin{aligned} \text{Hess}_\phi(\sigma) &= -i \int_{S \times_r \mathbb{R}^{1|1}/S} \langle a + \theta\eta, \nabla_{\partial_t}\eta + \theta(i\nabla_{\partial_t}^2 + \frac{1}{2}R\nabla_{\partial_t})a \rangle d\theta dt \\ &= - \int_{S \times_r \mathbb{R}} \langle a, (-\nabla_{\partial_t}^2 + \frac{i}{2}R\nabla_{\partial_t})a \rangle + \langle \eta, i\nabla_{\partial_t}\eta \rangle dt, \end{aligned}$$

as claimed.  $\square$

*Proof of Proposition 1.10.* To set up the computation, we pullback  $\Delta_{M/B}^{1|1}$  along the map  $u: \mathbb{R}_{>0} \times \underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M) \rightarrow \mathcal{L}_0^{1|1}(M)$ . By Lemma 2.7, on this pullback we have the identification

$$u^*(\Delta_{M/B}^{1|1})^{\text{ev}} = -\frac{d^2}{dt^2} \otimes \text{id}_{T(M/B)} + i\frac{d}{dt} \otimes R, \quad u^*(\Delta_{M/B}^{1|1})^{\text{odd}} = i\frac{d}{dt} \otimes \text{id}_{T(M/B)},$$

where now  $R$  is the  $\text{End}(p^*T(M/B))$ -valued function on  $\underline{\mathbf{SMfld}}(\mathbb{R}^{0|1}, M)$  associated to the curvature 2-form. We use the basis for functions on  $\mathbb{R}/r\mathbb{Z}$  given by  $\{e^{2\pi i n t/r}\}$ , yielding the  $\zeta$ -functions,

$$\begin{aligned} \zeta_\Delta^{\text{ev}}(s) &= \sum_{n \neq 0} \text{Tr} \left( \frac{4\pi^2 n^2}{r^2} \otimes \text{id}_{T(M/B)} + \frac{2\pi i n}{r} \otimes iR \right)^s \\ \zeta_\Delta^{\text{odd}}(s) &= \sum_{n \neq 0} \text{Tr} \left( -\frac{2\pi n}{r} \otimes \text{id}_{T(M/B)} \right)^s \end{aligned}$$

corresponding to the operators  $(\Delta_{M/B}^{1|1})^{\text{ev}}$  and  $(\Delta_{M/B}^{1|1})^{\text{odd}}$ , respectively. The contribution of  $\zeta_\Delta^{\text{odd}}$  to the  $\zeta$ -regularized super determinant only depends on the dimension  $n = \dim(M) - \dim(B)$ , and is  $(2\pi/r)^{n/2}$ , e.g., see Example 2 of [QHS93]. Binomial expansion in odd variables gives

$$\begin{aligned} \zeta_\Delta^{\text{ev}}(s) &= \sum_{n \neq 0} \text{Tr} \left( \text{id} - R \otimes \frac{r}{2\pi n} \right)^s \left( \frac{4\pi^2 n^2}{r^2} \right)^s \\ &= \sum_{n \neq 0} \sum_{k=0}^{\text{finite}} \text{Tr} \left( R^k \frac{s(s-1) \cdots (s-k+1)}{k!(2\pi n)^k} r^k \right) \left( \frac{4\pi^2 n^2}{r^2} \right)^s \end{aligned}$$

where the sum over  $k$  is finite because  $R$  is nilpotent. The  $k=0$  piece is another standard  $\zeta$ -regularized product, and contributes  $(2\pi/r)^{-n}$  to the  $\zeta$ -super determinant. For  $k > 1$ , we differentiate under the sum, and altogether we get

$$\text{sdet}_\zeta(\Delta_{M/B}^{1|1}) = (r/2\pi)^{n/2} \exp \left( \sum_{k=1}^{\infty} \frac{\text{Tr}(R^{2k}) r^{2k}}{2k(2\pi i)^{2k}} \zeta(2k) \right).$$

By Proposition 3.7, the normalization for which this function descends to  $\mathcal{L}_0^{1|1}(M)$  is therefore  $(r/2\pi)^{n/2}$ , which indeed corresponds to the reciprocal of the normalizing factor defining the Thom class. It remains to compare with the  $\hat{A}$ -form. In our cocycle model,

$$r^{2k} \text{Tr}(R)^{2k} = 2(2k)! \text{ph}_k(T(M/B)),$$

where  $\text{ph}_k$  denotes the  $4k^{\text{th}}$  component of the Pontryagin character as a function on  $\mathcal{L}_0^{1|1}(M)$ . Putting this together we get

$$\frac{\text{sdet}_\zeta(\Delta_{M/B}^{1|1})}{\overline{Z}^{\dim(B)-\dim(M)}} = \exp \left( \sum_{k=1}^{\infty} \frac{(2k)! \text{ph}_k(T(M/B))}{2k} \frac{2\zeta(2k)}{(2\pi i)^{2k}} \right)$$

which we identify as the  $\hat{A}$ -class of the family  $\pi: M \rightarrow B$  as a function on  $\mathcal{L}_0^{1|1}(X)$ .  $\square$

**3.5. An equality of differential pushforwards.** The constructions of  $\hat{A}(M/B)$  and the K-theoretic Mathai–Quillen form define cocycle pushforwards by same constructions at the level of cohomology classes in §1.3. As discussed there, the equality of these refinements of pushforwards boils down to an equality of differential cocycles,

$$(48) \quad \hat{A}(\nu)^{-1} = \hat{A}(M/B),$$

where the left-hand-side is the Riemann–Roch factor that modifies the Thom cocycle in de Rham cohomology, and the right hand side is the Riemann–Roch factor that modifies fiber integration of differential forms. For these cocycles to be equal, we require  $M \hookrightarrow \mathbb{R}^N$  to be a Riemannian embedding. Then, translating (48) into our construction of the Mathai–Quillen Thom form, we can rephrase the equality of Riemann–Roch factors (and hence, an index theorem over  $\mathbb{C}$ ) as follows.

**Proposition 3.13.** *Let  $\nu$  be the normal bundle to  $M \rightarrow B \times \mathbb{R}^N$  associated with an isometric embedding  $M \rightarrow \mathbb{R}^N$ . The functions on  $\mathcal{L}_0^{1|1}(M)$  defined by the normalized  $\zeta$ -super determinants of  $\mathcal{D}_\nu^{1|1}$  and  $\Delta_{M/B}^{1|1}$  are equal, which implies that the differential analytic and topological pushforwards agree for geometric families of oriented manifolds,*

$$\pi_!^{\text{an}} = \pi_!^{\text{top}}: \Gamma(\mathcal{L}_0^{1|1}(M); \omega^{k/2}) \rightarrow \Gamma(\mathcal{L}_0^{1|1}(B); \omega^{(k-n)/2}).$$

where  $n = \dim(M) - \dim(B)$ .

*Proof.* Since  $\text{sdet}_\zeta(\mathcal{D}_\nu^{1|1}) \cdot (r/2\pi)^{n/2}$  is the Riemann–Roch factor in the K-theoretic Mathai–Quillen form and  $\text{sdet}_\zeta(\Delta_{M/B}^{1|1}) \cdot (r/2\pi)^{-n/2}$  is the Riemann–Roch factor for fiberwise integration of forms, the equality of functions in the proposition is exactly the equality (48). With the Riemannian embedding fixed, the differential cocycles representing the Pontryagin characters of  $\nu$  and  $T(M/B)$  are inverse to one another:  $\text{ph}(\nu)^{-1} = \text{ph}(T(M/B))$ . This implies we get the claimed equality of cocycles and the result follows.  $\square$

#### 4. COCYCLE PUSHFORWARDS FOR COMPLEXIFIED TMF

This section closely parallels the previous one, with many of the proofs going through following the same arguments. We start by setting up the differential cocycle model for  $\text{TMF} \otimes \mathbb{C}$  in §4.1. Then we use analytic techniques to construct the elliptic Mathai–Quillen form from the family of operators  $\mathcal{D}_V^{2|1}$  (proving Theorem 1.4 in §4.3) and the Witten cocycle for a geometric family of oriented manifolds from the family of operators  $\Delta_{M/B}^{2|1}$  (proving Theorem 1.3 in §4.4). When  $V = \nu$  is the normal bundle for an embedding  $M \hookrightarrow B \times \mathbb{R}^N$ , we show there is an isomorphism of super determinant line bundles with section associated to these families of operators. This proves Theorem 1.5, which implies the index theorem for  $\text{TMF} \otimes \mathbb{C}$  as phrased in the introduction.

##### 4.1. Super double loop spaces and complexified TMF.

**Definition 4.1.** The 2|1-dimensional rigid conformal model geometry takes  $\mathbb{R}^{2|1}$  as its model space and the super group  $\mathbb{E}^{2|1} \rtimes \mathbb{C}^\times$  as isometry group, where  $\mathbb{E}^{2|1}$  is  $\mathbb{R}^{2|1}$  as a supermanifold with multiplication

$$(z, \bar{z}, \theta) \cdot (z', \bar{z}', \theta') = (z + z', \bar{z} + \bar{z}' + \theta\theta', \theta + \theta'), \quad (z, \bar{z}, \theta), (z', \bar{z}', \theta') \in \mathbb{R}^{2|1}(S),$$

and the semidirect product  $\mathbb{E}^{2|1} \rtimes \mathbb{C}^\times$  comes from the action  $(\mu, \bar{\mu}) \cdot (z, \bar{z}, \theta) = (\mu^2 z, \bar{\mu}^2 \bar{z}, \mu\theta)$ , for  $(z, \bar{z}, \theta) \in \mathbb{E}^{2|1}(S)$  and  $(\mu, \bar{\mu}) \in \mathbb{C}^\times(S)$ . We take the obvious left action of  $\mathbb{E}^{2|1} \rtimes \mathbb{C}^\times$  on  $\mathbb{R}^{2|1}$ . The Lie algebra of left-invariant vector fields on  $\mathbb{E}^{2|1}$  has a pair of commuting generators denoted  $\partial_z$  and  $D := \partial_\theta + \theta\partial_{\bar{z}}$  with  $D^2 = \partial_{\bar{z}}$ .

A family of 2-dimensional (oriented) lattices is an  $S$ -family of homomorphisms  $\Lambda: S \times \mathbb{Z}^2 \rightarrow S \times \mathbb{R}^2$  such that the ratio of the images of  $S \times \{1, 0\}$  and  $S \times \{0, 1\}$  under  $\Lambda: S \times \mathbb{Z}^2 \rightarrow S \times \mathbb{R}^2 \cong S \times \mathbb{C}$  are in  $\mathfrak{h} \subset \mathbb{C}$ . Let  $L$  denote the manifold whose  $S$ -points are oriented lattices; note that  $L \cong \mathbb{C}^\times \times \mathfrak{h}$ . Through the inclusion of groups  $\mathbb{E}^2 \subset \mathbb{E}^{2|1}$ , an  $S$ -family of lattices defines a family of *super tori* via the quotient  $S \times \mathbb{R}^{2|1}/\Lambda =: S \times_\Lambda \mathbb{R}^{2|1}$ .

**Definition 4.2.** The *super double loop stack* of  $M$ , denoted  $\mathcal{L}^{2|1}(M)$ , has as objects over  $S$  pairs  $(\Lambda, \phi)$  where  $\Lambda \in L(S)$  determines a family of super tori  $S \times_{\Lambda} \mathbb{R}^{2|1}$  and  $\phi: S \times_{\Lambda} \mathbb{R}^{2|1} \rightarrow M$  is a map. Morphisms between these objects over  $S$  consist of commuting triangles

$$(49) \quad \begin{array}{ccc} S \times_{\Lambda} \mathbb{R}^{2|1} & \xrightarrow{\cong} & S \times_{\Lambda'} \mathbb{R}^{2|1} \\ \searrow \phi & & \swarrow \phi' \\ & M & \end{array}$$

where the horizontal arrow is an isomorphism of  $S$ -families of rigid conformal super manifolds. The stack of *constant super tori*, denoted  $\mathcal{L}_0^{2|1}(M)$ , is the full substack for which  $\phi$  is invariant under the translational action of tori, i.e.,  $(\Lambda, \phi)$  is an  $S$ -point of  $\mathcal{L}_0^{2|1}(M)$  if for all families of isometries associated with sections of the bundle of groups  $S \times_{\Lambda} \mathbb{E}^2 \rightarrow S$ , the triangle (49) commutes with  $\Lambda = \Lambda'$  and  $\phi = \phi'$ .

Define a morphism of stacks  $\mathcal{L}^{2|1}(\text{pt}) \rightarrow \text{pt} // \mathbb{C}^{\times}$  that sends all objects over  $S$  to  $\text{pt}$  and to an isomorphism  $S \times_{\Lambda} \mathbb{R}^{1|1} \rightarrow S \times_{\Lambda'} \mathbb{R}^{2|1}$  associates the map  $S \rightarrow \mathbb{C}^{\times}$  that records the dilation factor on tori over  $S$ , i.e., is the component of the  $S$ -family of isometries between super tori associated with this dilation factor. There is a canonical odd line bundle over  $\text{pt} // \mathbb{C}^{\times}$  associated with the isomorphism of groups,  $\mathbb{C}^{\times} \cong \text{GL}(\mathbb{C}^{0|1})$ .

**Definition 4.3.** Define line bundles  $\omega^{m/2}$  over  $\mathcal{L}_0^{2|1}(M)$  as the pullback of the  $m$ th tensor power of the canonical odd line over  $\text{pt} // \mathbb{C}^{\times}$  along the composition

$$\mathcal{L}_0^{2|1}(M) \rightarrow \mathcal{L}_0^{2|1}(\text{pt}) \cong \mathcal{L}^{2|1}(\text{pt}) \rightarrow \text{pt} // \mathbb{C}^{\times}.$$

**4.2. An atlas for  $\mathcal{L}_0^{2|1}(M)$  and its groupoid presentation.** The proof of the following is identical to the 1|1-dimensional case.

**Lemma 4.4.** A map  $\phi$  is invariant under the translational action of tori if and only if it factors through the map  $S \times_{\Lambda} \mathbb{R}^{2|1} \rightarrow S \times \mathbb{R}^{0|1}$  induced by the projection  $\mathbb{R}^{2|1} \rightarrow \mathbb{R}^{0|1}$ .

For an  $S$ -point of the constant super tori, this factorization property means  $\phi$  is determined by a map  $\phi_0: S \times \mathbb{R}^{0|1} \rightarrow M$ . This gives an atlas,

$$(50) \quad u: L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \rightarrow \mathcal{L}_0^{2|1}(M),$$

which determines a groupoid presentation whose objects are  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$ . To compute the supermanifold of morphisms, we use that super tori are a quotient of  $\mathbb{R}^{2|1}$ , giving an exact sequence

$$(51) \quad \mathbb{Z}^2 \xrightarrow{\Lambda} \underline{\text{Iso}}(\mathbb{R}^{2|1}) \rightarrow \underline{\text{Iso}}(\mathbb{R}^{2|1}/\Lambda).$$

The action by translations  $\mathbb{E}^{2|1} < \underline{\text{Iso}}(\mathbb{R}^{1|1})$  leaves the lattice  $\Lambda \subset \mathbb{R}^{2|1}$  unchanged, whereas dilations  $(\mu, \bar{\mu}) \in \mathbb{C}^{\times}$  dilate the lattice, and  $\text{SL}_2(\mathbb{Z})$  acts by changing the basis of  $\Lambda$ . We summarize this discussion by the following.

**Proposition 4.5.** There is an essentially surjective full morphism of stacks,

$$L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) // (\underline{\text{Iso}}(\mathbb{R}^{2|1}) \times \text{SL}_2(\mathbb{Z})) \rightarrow \mathcal{L}_0^{2|1}(M),$$

where the  $\underline{\text{Iso}}(\mathbb{R}^{2|1})$ -action on  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  is through the homomorphism  $\underline{\text{Iso}}(\mathbb{R}^{2|1}) \rightarrow \mathbb{E}^{0|1} \rtimes \mathbb{C}^{\times}$  followed by the precomposition action on  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$ , and on  $L$  through the projection to  $\mathbb{C}^{\times}$  followed by the dilation action. The action by  $\text{SL}_2(\mathbb{Z})$  is trivial on  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$ , and changes the basis of the lattice. This induces an equivalence of stacks,

$$\left( \begin{array}{c} (\underline{\text{Iso}}(\mathbb{R}^{2|1}) \times L) / \mathbb{Z}^2 \times \text{SL}_2(\mathbb{Z}) \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \\ \downarrow \downarrow \\ L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \end{array} \right) \xrightarrow{\sim} \mathcal{L}_0^{2|1}(M),$$

where the quotient by  $\mathbb{Z}^2$  is the quotient by the fiberwise kernel of (51).

The homomorphism

$$\underline{\text{Iso}}(\mathbb{R}^{2|1}) \cong \mathbb{E}^{2|1} \rtimes \mathbb{C}^\times \twoheadrightarrow \mathbb{C}^\times \cong \text{GL}(\mathbb{C}^{0|1})$$

gives a functor from the groupoid in Proposition 4.5 to  $\text{pt}/\mathbb{C}^\times$ , and the pullback of the canonical odd line bundle is isomorphic to the pullback of  $\omega^{1/2}$ . This allows us to compute sections of  $\omega^{m/2}$  in terms of functions on  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  with transformation properties. Recall that a weak *Maass form* is a function on lattices that transforms as a weak modular form, but need not be holomorphic. Let  $\text{MaF}$  denote the graded ring of Maass forms.

**Proposition 4.6.** *Sections of  $\omega^{k/2}$  are spanned by functions on  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  of the form*

$$(52) \quad \text{vol}^{j/2} F \otimes f \in C^\infty(L) \otimes \Omega_{\text{cl}}^j(M) \subset C^\infty(\mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M))$$

where  $\text{vol}$  is the function on  $L$  that reads off the volume of the torus with a given lattice,  $F \in \text{MaF}^i \subset \mathcal{O}(L)$  defines a weak Maass form of weight  $-i/2$  and  $i + j = k$ . In particular, we get isomorphisms

$$\mathcal{O}(\mathcal{L}_0^{2|1}(M); \omega^{k/2}) \cong \bigoplus_{i+j=k} \Omega_{\text{cl}}^j(M; \text{MaF}^i)$$

compatible with the multiplications.

*Proof.* We identify a section of  $\omega^{k/2}$  with a function on  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  invariant under the  $\mathbb{E}^{2|1} \times \text{SL}_2(\mathbb{Z})$ -action and equivariant for the  $\mathbb{C}^\times$ -action. The  $\mathbb{E}^{2|1}$ -action is generated by the de Rham operator on  $C^\infty(\underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)) \cong \Omega^\bullet(M)$ , and the  $\text{SL}_2(\mathbb{Z})$ -action on  $L$  is by changing the basis of a lattice. So the invariant functions are  $C^\infty(L)^{\text{SL}_2(\mathbb{Z})} \otimes \Omega_{\text{cl}}^\bullet(M)$ . The  $\mathbb{C}^\times$ -action is diagonal on  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$ , reading off the weight of a Maass form  $F \in C^\infty(L)$  and the degree of a differential form  $f \in \Omega_{\text{cl}}^k(M)$ . This gives the sections claimed in the proposition.  $\square$

We give a low-brow definition of holomorphic section of  $\omega^{\bullet/2}$ ; a more geometric discussion involving dilations of super tori is in [BE13, §3].

**Definition 4.7.** *Holomorphic sections of  $\omega^{\bullet/2}$  are by spanned by those sections that when pulled back to the atlas  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  are of the form*

$$\text{vol}^{j/2} F \otimes f \in C^\infty(L) \otimes \Omega_{\text{cl}}^j(M) \subset C^\infty(\mathbb{R}_{>0} \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M))$$

for  $F \in \mathcal{O}(L) \cong \mathcal{O}(\mathbb{C}^\times \times \mathfrak{h})$  a holomorphic function.

*Proof of Theorem 1.1.* Directly from Proposition 4.6 and Definition 4.7, we see that holomorphic sections of  $\omega^{\bullet/2}$  are identified with closed differential forms valued in modular forms. Naturality of  $M \mapsto \mathcal{O}(\mathcal{L}_0^{2|1}(M); \omega^{\bullet/2})$  in  $M$  makes this an isomorphism of presheaves. But since the target of the isomorphism is itself a sheaf, we get the claimed isomorphism of sheaves.  $\square$

**4.3. The elliptic Mathai–Quillen form.** To construct a Thom cocycle, we again consider the digram (39), and by pulling back the  $C^\infty(S)$ -module  $\Gamma(S \times_{\Lambda} \mathbb{R}^{2|1}, \phi^* p^* V)$  along isometries of super tori we obtain a vector bundle  $\mathcal{F}^{d|1}(V)$  over the stack  $\mathcal{L}_0^{2|1}(M)$ . The usual formula determines a function on sections

$$\mathcal{S}_{\text{MQ}}(\Psi) = \frac{1}{2} \int_{S \times_{\Lambda} \mathbb{R}^{2|1}/S} \left( \langle \Psi, \nabla_D \Psi \rangle - \frac{i}{\sqrt{\text{vol}}} \langle \Psi, \phi^* \mathbf{x} \rangle \right) d\theta \frac{i}{2} d\bar{z} dz$$

where  $D = \partial_\theta + \theta \partial_{\bar{z}}$  and we rescale the second term by the volume of the torus in question. As in the 1|1-dimensional case, this guarantees that  $\mathcal{S}_{\text{MQ}}$  is invariant under rigid conformal rescalings. We define component fields of  $\Psi$  and  $\phi^* \mathbf{x}$  by the same formulas (40) as before, but with the new meaning for  $\Psi$  and  $D$ . The following is proved in an identical fashion as in the 1|1-dimensional case.



**Proposition 4.8.** *Evaluated on the atlas  $L \times \mathbf{SMfld}(\mathbb{R}^{0|1}, V)$ , the functional  $\mathcal{S}_{\text{MQ}}$  takes the form*

$$\mathcal{S}_{\text{MQ}}(\psi_1, \psi_0) = \frac{1}{2} \int_{S \times_{\Lambda} \mathbb{R}^2} \left( \langle \psi_0, \psi_0 - \frac{i}{\sqrt{\text{vol}}} \mathbf{x} \rangle + \langle \psi_1, (-i\nabla_{\partial_{\bar{z}}} + F)\psi_1 + \frac{i}{\sqrt{\text{vol}}} \nabla \mathbf{x} \rangle \right) d\bar{z} dz$$

where as before we regard  $\mathbf{x}, \nabla \mathbf{x}$  and  $F$  as section- or endomorphism valued functions on  $\mathbf{SMfld}(\mathbb{R}^{0|1}, V)$ .

Via (21), we make sense out of  $\int e^{-\mathcal{S}_{\text{MQ}}(\Psi)} d\Psi$  by pulling back  $\mathcal{F}^{2|1}(V)$  to the atlas (50) and splitting the space of sections into a finite-dimensional piece  $\mathcal{F}_0^{2|1}(V)$  that contributes the ordinary Thom class, and an infinite-dimensional piece  $\mathcal{F}_{\perp}^{2|1}(V)$  that will turn out to contribute the inverse Witten class of  $V$ . The fiberwise volume form on  $S \times_{\Lambda} \mathbb{R}^2$  defines pairings on sections analogous to (41) in the 1|1-dimensional case.

**Definition 4.9.** Define the bundle of *zero modes*  $\mathcal{F}_0^{2|1}(V) \subset p^* \mathcal{F}^{2|1}(V)$  as the subbundle whose  $S$ -points have  $\nabla_{\partial_{\bar{z}}} \Psi = 0$ . Let  $\mathcal{F}_{\perp}^{2|1}(V)$  denote the orthogonal complement to  $\mathcal{F}_0^{2|1}(V)$  in  $p^* \mathcal{F}^{2|1}(V)$ .

As in the 1|1-dimensional case,  $\mathcal{F}_0^{2|1}(V)$  is a finite-dimensional vector bundle over  $\mathcal{L}_0^{2|1}(V)$ : holomorphic sections on tori are the same as constant sections. The restriction of  $\mathcal{S}_{\text{MQ}}$  to  $\mathcal{F}_0^{2|1}(V)$  coincides with the image of the Mathai–Quillen form in ordinary cohomology.

*Proof of Lemma 1.8 when  $d = 2$ .* The argument is identical to the  $d = 1$  case, replacing  $r$  by  $\text{vol}$ .  $\square$

Next we analyze  $\mathcal{S}_{\text{MQ}}$  on  $\mathcal{F}_{\perp}^{2|1}(V)$ , verifying (18) and (20)

**Lemma 4.10.** *For sections  $\Psi$  of  $\mathcal{F}_{\perp}^{2|1}(V)$ , we have*

$$(53) \quad \mathcal{S}_{\text{MQ}}(\Psi) = \int_{S \times_r \mathbb{R}^{1|1}} \langle \Psi, \mathcal{D}_V^{2|1} \Psi \rangle,$$

where  $\mathcal{D}_V^{2|1}$  is the restriction of  $\nabla_D$ . In components,

$$(54) \quad \mathcal{S}_{\text{MQ}}(\psi_1, \psi_0) = \int_{S \times_r \mathbb{R}} (\langle \psi_0, \psi_0 \rangle + \langle \psi_1, (-i\nabla_{\partial_{\bar{z}}} + F)\psi_1 \rangle) dt$$

for  $-i\nabla_{\partial_{\bar{z}}} + F$  a family of invertible operators.

*Proof.* The argument is identical to the proof of Lemma 3.10 in the 1|1-dimensional case.  $\square$

It remains to compute the  $\zeta$ -super determinant of  $\mathcal{D}_V^{2|1}$  on the cover  $L \times \mathbf{SMfld}(\mathbb{R}^{0|1}, V)$  of  $\mathcal{L}_0^{2|1}(V)$ , but there is a subtlety: this  $\zeta$ -super determinant is associated to a conditionally convergent series. An  $S^1$ -action on tori gotten from a choice of meridian  $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$  determines an  $\mathbb{Z}$ -grading whose  $n^{\text{th}}$  graded piece is  $\mathcal{F}_{n, \bullet}^{2|1}$  in the notation of (17). This  $\mathbb{Z}$ -grading fixes an order of summation for the  $\zeta$ -function associated to  $\mathcal{D}_V^{2|1}$  (see §A.5), and this fixes the value of the conditionally convergent series. Let  $\mathcal{D}_n$  denote the restriction of  $\mathcal{D}_V^{2|1}$  to  $\mathcal{F}_{n, \bullet}^{2|1}$  for  $n \neq 0$  and  $\oplus_{m \neq 0} \mathcal{F}_{0, m}^{2|1}$  for  $n = 0$ . Then define the  $\zeta$ -function

$$(55) \quad \zeta_{\mathcal{D}}^{\text{ren}}(s) = \sum_n \zeta_{\mathcal{D}_n}(s)$$

where the summation is ordered: first for the  $\zeta$ -functions  $\zeta_{\mathcal{D}_n}(s)$ , and then over  $n$ .

**Definition 4.11.** Let  $\text{sdet}_{\zeta}^{\text{ren}}(\mathcal{D}_V^{2|1})$  denote the  $\zeta$ -super determinant associated with the ordered sum (55).

*Proof of 1.9.* By definition, the  $\zeta$ -super determinant is a ratio of  $\zeta$ -super determinants applied to operators acting on even and odd section on this pull back. The relevant operator on the even sections  $\psi_0$  is the identity, so this contributes 1 to the  $\zeta$ -super determinant.

To compute the contribution from odd sections, choose a basis for functions on tori

$$(56) \quad f_{n,m}(z, \bar{z}) := \exp\left(\frac{\pi}{\text{vol}}(-z(n\bar{\ell} + m\bar{\ell}') + \bar{z}(n\ell + m\ell'))\right), \quad (m, n) \in \mathbb{Z} \times \mathbb{Z}.$$

Sections  $\psi_1$  are then of the form  $f_{n,m} \otimes v$  for  $v$  a section of  $\text{IIV}$  pulled back to  $\underline{\text{SMfld}}(\mathbb{R}^{0|1}, V)$ . We get the  $\zeta$ -function

$$(57) \quad \zeta_{\mathcal{D}}(s) = \sum_{(n,m) \in \mathbb{Z}_*^2} \text{Tr} \left( \frac{\pi}{\text{vol}}(m\ell + n\ell') \otimes \text{Id}_{TX} + \text{id} \otimes F \right)^s,$$

where  $\mathbb{Z}_*^2 = \mathbb{Z}^2 \setminus (0, 0)$ . Applying binomial expansion

$$\begin{aligned} \zeta_{\mathcal{D}}(s) &= \sum_{(n,m) \in \mathbb{Z}_*^2} \text{Tr} \left( \left( \text{Id}_{TX} + \frac{\text{vol}}{2\pi}(m\ell + n\ell')^{-1} \otimes F \right)^s \left( \frac{\pi}{\text{vol}}(m\ell + n\ell') \right)^s \right) \\ &= \sum_n \sum_{k=0}^{\text{finite}} \text{Tr} \left[ \left( \text{vol}^k \frac{s(s-1) \cdots (s-k+1)}{k!(2\pi)^k (m\ell + n\ell')^k} \otimes F^k \right) \left( \frac{\pi(m\ell + n\ell')}{\text{vol}} \right)^s \right] \end{aligned}$$

where the sum is finite because  $F$  is nilpotent. By the definition of  $\text{sdet}_{\zeta}^{\text{ren}}(\mathcal{D}_V^{2|1})$ , we evaluate  $\zeta'_{\mathcal{D}}(0)$  by first summing over  $m$ , and then over  $n$ , and use the symbol  $\sum_{(n,m) \in \mathbb{Z}_*^2}^{\text{ren}}$  to denote this choice of ordering. Using that odd powers of  $F$  have trace zero, we get

$$\begin{aligned} \text{sdet}_{\zeta}^{\text{ren}}(\mathcal{D}_V^{2|1}) &= Z^n \exp \left( \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}_*^2}^{\text{ren}} \sum_{k=0}^{\text{finite}} \text{Tr} \left( \frac{(-1)^{k-1}}{k} \frac{\text{vol}^k}{(2\pi)^k} (m\ell + n\ell')^{-k} \otimes F^k \right) \right) \\ (58) \quad &= Z^n \exp \left( - \sum_{k=1}^{\infty} \frac{\text{vol}^{2k} E_{2k}}{2k(2\pi)^{2k}} \text{Tr}(F^{2k}) \right), \end{aligned}$$

where traces of odd powers of  $F$  vanish and

$$(59) \quad Z = \left( \frac{2\pi}{\ell/\ell'} \right)^{1/2} \prod_{k=1}^{\infty} (1 - e^{2\pi i k(\ell/\ell')})$$

is the contribution from the  $k = 0$  term, coming from a known  $\zeta$ -regularized product (e.g., see [QHS93] Example 11), and is very nearly the Dedekind  $\eta$ -function. For  $k = 2$ , the summation is convergent with respect to the ordering of  $(m, n) \in \mathbb{Z}_*^2$ , and for  $k > 2$  the series converges absolutely without a choice of ordering.

In our cocycle model, we have

$$\text{vol}^{2k} \text{Tr}(F^{2k}) = 2(2k)! \text{ph}_k(V)$$

and so

$$\frac{\text{sdet}_{\zeta}^{\text{ren}}(\mathcal{D}_V^{2|1})}{Z^n} = \exp \left( \sum_{k \geq 1} \frac{(2k)! \text{ph}_k(V)}{2k(2\pi i)^{2k}} E_{2k} \right) \in C^\infty(L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, V))$$

is the non-modular Witten class of  $V$  as a function on  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, V)$ .

However, this function does *not* automatically descend to one on  $\mathcal{L}_0^{2|1}(V)$ : the 2<sup>nd</sup> Eisenstein series is not a modular form (see §A.4), and its transformation properties define a cocycle for a line bundle on  $\mathcal{L}_0^{2|1}(V)$  whose pullback to  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, V)$  trivializes. Given a rational string structure  $H$  with  $p_1(V) = dH$ , we can consider the concordance of sections

$$(60) \quad \exp \left( \frac{\text{vol}^2 d(\lambda H)}{2(2\pi i)^2} E_2 + \sum_{k \geq 2} \frac{\text{vol}^{2k} \text{Tr}(F^{2k})}{2k(2\pi i)^{2k}} E_{2k} \right) \in C^\infty(L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, V \times \mathbb{R}))$$

whose transformation properties for isomorphisms in  $\mathcal{L}_0^{2|1}(V \times \mathbb{R})$  determine a concordance of line bundles. In the target of this concordance, the term involving the 2<sup>nd</sup> Eisenstein series is eliminated, thereby giving a function on  $L \times \mathbf{SMfld}(\mathbb{R}^{0|1}, V)$  that does descend to  $\mathcal{L}_0^{2|1}(V)$ . By construction, the section of the trivial bundle is  $\text{Wit}_H(V)^{-1}$ , pulled back from  $\mathcal{L}_0^{2|1}(M)$  along the projection  $V \rightarrow M$ .  $\square$

**4.4. The Witten class of a geometric family.** To define the operators  $\Delta_{M/B}^{2|1}$ , first we specify the vector bundle on which they act. For  $M \rightarrow B$  a geometric family of oriented manifolds, define an infinite-rank vector bundle  $\mathcal{T}^{2|1}(M/B) \rightarrow \mathcal{L}^{2|1}(M)$  whose fiber at an  $S$ -point is the  $C^\infty(S)$ -module  $\Gamma(S \times_\Lambda \mathbb{R}^{2|1}, \phi^*T(M/B))$ . The metric and connection on  $T(M/B)$  pull back to these spaces of sections, and the fiberwise volume form on  $S \times_\Lambda \mathbb{R}^2$  gives a pairing on sections of  $\mathcal{T}^{2|1}(M/B)$  at each  $S$ -point. We define Taylor components identically to (45), where now  $D = \partial_\theta + \theta \partial_{\bar{z}}$ . Similarly, we obtain a pairing on sections using the metric on  $T(M/B)$  and the volume forms on tori.

**Definition 4.12.** Define the vector bundle  $\mathcal{N}^{2|1}(M/B) \subset \mathcal{T}^{2|1}(M/B)|_{\mathcal{L}_0^{2|1}(X)}$  over  $\mathcal{L}_0^{2|1}(X)$  as having  $S$ -points sections in the orthogonal complement of the constant sections, where a section  $\sigma$  is constant if  $\nabla_{\partial_{\bar{z}}} \sigma = 0$ . We use the notation  $\Gamma_0(S \times_\Lambda \mathbb{R}^{2|1}, \phi^*T(M/B)) \subset \Gamma(S \times_\Lambda \mathbb{R}^{2|1}, \phi^*T(M/B))$  to denote this orthogonal complement at an  $S$ -point  $(\Lambda, \phi)$ .

Define a function on sections of  $\mathcal{N}^{2|1}(M/B)$  by

$$(61) \quad \text{Hess}_\phi(\sigma) := \int_{S \times_\Lambda \mathbb{R}^{2|1}/S} \langle \sigma, \nabla_{\partial_{\bar{z}}} \nabla_D \sigma \rangle d\theta \frac{i}{2} d\bar{z} dz, \quad \sigma \in \Gamma_0(S \times_\Lambda \mathbb{R}^{2|1}, \phi^*T(M/B))$$

where  $D = \partial_\theta + \theta \partial_{\bar{z}}$  is the right-invariant vector field on the family  $S \times \mathbb{R}^{2|1}$ , and the integral is the Berezinian integral over the fibers of the projection  $S \times_\Lambda \mathbb{R}^{2|1} \rightarrow S$ . Since it is built out of right-invariant vector fields on  $\mathbb{R}^{2|1}$ , the function Hess is automatically invariant under the left action of isometries, so it defines a function on the stack as claimed. We use the notation  $\Delta_{M/B}^{2|1} = \nabla_{\partial_{\bar{z}}} \nabla_D$  to emphasize the dependence of this family of operators on  $\pi: M \rightarrow B$ . A component-form version of this function will facilitate computations.

**Lemma 4.13.** *Taylor expanding  $\sigma$  and performing the Berezin integral in (23),*

$$\begin{aligned} \text{Hess}_\phi(\sigma) &= \text{Hess}_\phi(a, \eta) = \int_{S \times_\Lambda \mathbb{R}^{2|1}/S} \langle (\Delta_{M/B}^{2|1})^{\text{ev}} a, a \rangle + \langle (\Delta_{M/B}^{2|1})^{\text{odd}} \eta, \eta \rangle \frac{i}{2} d\bar{z} dz, \\ (\Delta_{M/B}^{2|1})^{\text{ev}} &:= -\nabla_{\partial_{\bar{z}}} \nabla_{\partial_z} + \frac{1}{2} R \nabla_{\partial_z}, \quad (\Delta_{M/B}^{2|1})^{\text{odd}} = \nabla_{\partial_z} \end{aligned}$$

where  $R := \phi^*R(D, D)$  is the  $\text{End}(\phi^*T(M/B))$ -valued function on  $S \times_\Lambda \mathbb{R}^{2|1}$  determined by the curvature 2-form of the Levi–Civita connection.

The proof is identical to the 1|1-dimensional case, Lemma 3.12. We endow  $\mathcal{N}^{2|1}(M/B)$  with an  $\mathbb{Z}$ -grading whose  $n^{\text{th}}$  graded piece is  $\mathcal{N}_{n, \bullet}^{2|1}(M/B)$  in the notation of (26). This grading specifies an ordered  $\zeta$ -function in complete analogy to (55); let  $\text{sdet}_\zeta^{\text{ren}}(\Delta_{M/B}^{2|1})$  be the  $\zeta$ -super determinant associated with this ordered sum.

*Proof of Theorem 1.3.* To set up the computation, we pullback  $\Delta_{M/B}^{2|1}$  along the map  $u: L \times \mathbf{SMfld}(\mathbb{R}^{0|1}, M) \rightarrow \mathcal{L}_0^{2|1}(M)$ . By Lemma 2.7, on this pullback we have identification

$$u^*(\Delta_{M/B}^{2|1})^{\text{ev}} = -\frac{\partial^2}{\partial z \partial \bar{z}} \otimes \text{id}_{T(M/B)} + i \frac{\partial}{\partial z} \otimes R, \quad u^*(\Delta_{M/B}^{2|1})^{\text{odd}} = i \frac{\partial}{\partial z} \otimes \text{id}_{T(M/B)},$$

where now  $R$  is the  $\text{End}(p^*T(M/B))$ -valued function on  $\mathbf{SMfld}(\mathbb{R}^{0|1}, M)$  associated to the curvature 2-form. As in the construction of the Mathai–Quillen form, the  $n^{\text{th}}$  graded subspace of the space of sections is spanned by  $\{f_{n, \bullet} \otimes v\}$  for  $v \in \Gamma(S \times \mathbb{R}^{0|1}, \phi_0^*T(M/B))$  and  $f_{n, m}$  as in (56).

We form the  $\zeta$ -functions for the operators  $u^*(\Delta_{M/B}^{2|1})^{\text{ev}}$  and  $u^*(\Delta_{M/B}^{2|1})^{\text{odd}}$

$$\begin{aligned}\zeta_{\Delta}^{\text{ev}}(s) &= \sum_{(m,n) \in \mathbb{Z}_*^2} \text{Tr} \left( \frac{\pi^2}{\text{vol}^2} |m\ell + n\ell'|^2 \otimes \text{Id}_{T(M/B)} + \frac{\pi}{\text{vol}} (m\bar{\ell} + n\bar{\ell}') \otimes R \right)^s \\ \zeta_{\Delta}^{\text{odd}}(s) &= \sum_{(m,n) \in \mathbb{Z}^2} \text{Tr} \left( \frac{\pi}{\text{vol}} (m\bar{\ell} + n\bar{\ell}') \otimes \text{Id}_{T(M/B)} \right)^s\end{aligned}$$

corresponding to the operators  $u^*(\Delta_{M/B}^{2|1})^{\text{ev}}$  and  $u^*(\Delta_{M/B}^{2|1})^{\text{odd}}$  acting on each  $\mathbb{Z}$ -graded subspace of sections. The contribution from operators acting on odd sections can be computed following Example 11 in [QHS93] yielding the conjugate of  $Z^n$  in the notation of (59). For the operators on even sections, we take the binomial expansion,

$$\begin{aligned}\zeta_{\Delta}^{\text{ev}}(s) &= \sum_{(m,n) \in \mathbb{Z}_*^2}^{\text{ren}} \text{Tr} \left( \left( \text{Id}_{TX} + \frac{\text{vol}}{2\pi} (m\ell + n\ell')^{-1} \otimes R \right)^s \left( \frac{\pi^2}{\text{vol}^2} |m\ell + n\ell'|^2 \right)^s \right) \\ &= \sum_{(m,n) \in \mathbb{Z}_*^2}^{\text{ren}} \sum_{k=0}^{\text{finite}} \text{Tr} \left[ \left( \text{vol}^k \frac{s(s-1) \cdots (s-k+1)}{k!(2\pi)^k (m\ell + n\ell')^k} \otimes R^k \right) \left( \frac{\pi |m\ell + n\ell'|}{\text{vol}} \right)^{2s} \right]\end{aligned}$$

where, as above,  $\sum_{(m,n) \in \mathbb{Z}_*^2}^{\text{ren}}$  denotes the sum first over  $m$  and then over  $n$  (not both zero). The sum over  $k$  is finite because  $R$  is nilpotent. For  $k > 0$ , we can differentiate under the sum to obtain the following contribution to  $\zeta'(0)$ :

$$\sum_{(n,m) \in \mathbb{Z}_*^2}^{\text{ren}} \sum_{k=1}^{\text{finite}} \text{Tr} \left( \frac{(-1)^{k-1}}{k} \frac{\text{vol}^k}{(2\pi)^k} (m\ell + n\ell')^{-k} \otimes R^k \right) = - \sum_{k=2}^{\infty} \frac{\text{vol}^{2k} E_{2k}}{2k(2\pi)^{2k}} \text{Tr}(R^{2k})$$

where we have used that odd powers of  $R$  have trace zero. For  $k = 0$ , we obtain yet another standard  $\zeta$ -regularized determinant (see Example 9 in [QHS93]) which contributes  $2\pi|\eta(\ell, \ell')|^2$  for  $\eta$  the Dedekind  $\eta$ -function. So altogether we have

$$\frac{\text{sdet}_{\zeta}(\Delta_{M/B}^{2|1})}{Z^{-n}} = \exp \left( \sum_{k \geq 1}^{\infty} \frac{\text{vol}^{2k} \text{Tr}(R^{2k})}{4k(2\pi i)^{2k}} E_{2k} \right),$$

as a function on  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$ .

As in the case of the elliptic Mathai–Quillen form, this function does *not* automatically descend to one on  $\mathcal{L}_0^{2|1}(M)$ , again, owing to the 2<sup>nd</sup> Eisenstein series. Its transformation properties define a cocycle for a line bundle on  $\mathcal{L}_0^{2|1}(M)$  whose pullback to  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  trivializes. However, given a rational string structure  $H$  with  $dH = p_1(T(M/B))$ , we can consider the concordance of sections

$$(62) \exp \left( \frac{\text{vol}^2 d(\lambda H)}{2(2\pi i)^2} E_2 + \sum_{k \geq 2}^{\infty} \frac{\text{vol}^{2k} \text{Tr}(F^{2k})}{2k(2\pi i)^{2k}} E_{2k} \right) \in C^{\infty}(L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M \times \mathbb{R}))$$

whose transformation properties determine a concordance of line bundles on  $\mathcal{L}_0^{2|1}(M)$ . In the target of this concordance, the term involving the 2<sup>nd</sup> Eisenstein series is eliminated, thereby giving a function on  $L \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M)$  that does descend to  $\mathcal{L}_0^{2|1}(M)$ . By construction the section of the trivial bundle is the Witten class of  $T(M/B)$  with its chosen rational string structure.  $\square$

**4.5. An equality of differential pushforwards.** The constructions of  $\text{Wit}(M/B)$  and the elliptic Mathai–Quillen form define cocycle pushforwards by same constructions at the level of cohomology classes in §1.3. As discussed there, the equality of these differential pushforwards boils down to an equality of differential cocycles,

$$(63) \quad \text{Wit}(\nu)^{-1} = \text{Wit}(M/B),$$

where the left-hand-side is the Riemann–Roch factor that modifies the Thom cocycle in de Rham cohomology, and the right hand side is the Riemann–Roch factor that modifies integration of differential forms. For these cocycle to be equal, we require  $M \hookrightarrow \mathbb{R}^N$  to be a Riemannian embedding.

*Proof of Theorem 1.5.* Since  $\text{sdet}_\zeta^{\text{ren}}(\mathcal{D}_\nu^{2|1}) \cdot Z^{-n}$  is the Riemann–Roch factor in the elliptic Mathai–Quillen form and  $\text{sdet}_\zeta^{\text{ren}}(\Delta_{M/B}^{2|1}) \cdot Z^n$  is the Riemann–Roch factor for the fiber integration of differential forms, the equality of sections in the Theorem is exactly the equality (63). With the Riemannian embedding fixed, the Pontryagin characters are inverse to each other, guaranteeing the equality of these sections as functions on  $L \times \mathbf{SMfld}(\mathbb{R}^{0|1}, M)$ . Hence the line bundles over the stack  $\mathcal{L}_0^{2|1}(M)$  associated to the respective normalizations of these functions are canonically isomorphic and the super determinant sections are identified by this isomorphism.  $\square$

## APPENDIX A. BACKGROUND MISCELLANY

**A.1. Super manifolds and super stacks.** A  $k|l$ -dimensional super manifold is a locally ringed space whose structure sheaf is locally isomorphic to  $C^\infty(U) \otimes_{\mathbb{C}} \Lambda^\bullet(\mathbb{C}^l)$  as a super algebra over  $\mathbb{C}$  for  $U \subset \mathbb{R}^k$  an open submanifold. These are called *cs*-manifolds in [DM99]. Super manifolds and maps between them (as locally ringed spaces) form a category we denote by  $\mathbf{SMfld}$ . By M. Batchelor’s Theorem [Bat79], any super manifold  $N$  is isomorphic to  $(|N|, \Gamma(\Lambda^\bullet E^*))$  for  $E \rightarrow |N|$  a complex vector bundle over a smooth manifold  $|N|$ . We denote such a super manifold by  $\Pi E$ . The super manifold  $\mathbb{R}^{n|m}$  is the locally ringed space with structure sheaf  $C^\infty(\mathbb{R}^n) \otimes_{\mathbb{C}} \Lambda^\bullet(\mathbb{C}^m)$ , i.e.,  $\Pi \mathbb{R}^m$  for  $\underline{\mathbb{R}}^m \rightarrow \mathbb{R}^n$  the trivial bundle.

A *vector bundle* over a super manifold is a finitely generated projective module over the structure sheaf. In a slight abuse of terminology, we call elements of these modules *sections* of the vector bundle. Let  $\mathbf{SMfld}(N, T)$  denote the presheaf on supermanifolds whose value at  $S$  is the set  $\mathbf{SMfld}(S \times N, T)$ .

A *super stack* is a category fibered in groupoids over super manifolds satisfying descent with respect to surjective submersions of supermanifolds. We will often drop the modifier “super” when discussing super stacks. In practice, all of our stacks are *geometric*, meaning they admit an atlas and hence a Lie groupoid presentation. See [Blo08, Ler10] for a discussion of the non-super case, and [HKST11] for the super version.

**A.2. Model super geometries.** A *model (super) geometry* is determined by a model (super) space  $\mathbb{M}$  with the action of a (super) Lie group  $G$ . From this we obtain a category of  $(\mathbb{M}, G)$ -super manifolds fibered over the category of super manifolds. Objects over  $S \in \mathbf{SMfld}$  are bundles of super manifolds constructed by gluing open submanifolds of  $\mathbb{M}$  along (restrictions of) the action of  $G$  on  $\mathbb{M}$ . Isometries of  $(\mathbb{M}, G)$ -super manifolds are fiberwise diffeomorphisms over  $S$  that restrict locally to an action of  $G$  on  $\mathbb{M}$ ; see [HST10] Section 6.3 for details.

The main source of model super geometries are extensions of the super Euclidean geometries, which are themselves Wick-rotated versions of the standard super Poincaré geometries, e.g., see Freed [Fre99].

**Example A.1** (Super Euclidean geometries). Given data: (1) a real vector space  $V$  with inner product; (2) a complex spinor representation  $\Delta$  of  $\text{Spin}(V)$ ; (3) a  $\text{Spin}(V)$ -equivariant symmetric pairing  $\Gamma: \Delta \otimes \Delta \rightarrow V_{\mathbb{C}}$  we define the super group

$$(V \times \Pi\Delta) \times (V \times \Pi\Delta) \rightarrow (V \times \Pi\Delta), \quad (v, \sigma) \cdot (v', \sigma') = (v + v' + \Gamma(\sigma, \sigma'), \sigma + \sigma').$$

When  $\dim_{\mathbb{R}}(V) = d$  and  $\dim_{\mathbb{C}}(\Delta) = \delta$ , we write  $\mathbb{R}^{d|\delta} := V \times \Pi\Delta$  for the *super Euclidean space* that carries an action of a group of *super Euclidean translations* that we denote by  $\mathbb{E}^{d|\delta}$ . There is an exact sequence of groups,

$$0 \rightarrow V \rightarrow (V \times \Pi\Delta) \rightarrow \Pi\Delta \rightarrow 0$$

so that the ordinary translations of  $V$  form a subgroup of  $\mathbb{E}^{d|\delta}$ . We also have an action of  $\text{Spin}(V)$  on  $V \times \Pi\Delta$  via the spinor representation on  $\Delta$  and through the homomorphism  $\text{Spin}(V) \rightarrow \text{SO}(V)$  on  $V$ . This defines a super group  $\underline{\text{Iso}}(\mathbb{R}^{d|\delta}) := \mathbb{E}^{d|\delta} \rtimes \text{Spin}(V)$ , the *super Euclidean isometry group*. The pair  $\mathbb{R}^{d|\delta}$  and  $\underline{\text{Iso}}(\mathbb{R}^{d|\delta})$  define a *super Euclidean geometry*.

**Notation A.2.** To distinguish between translation groups and the super manifolds on which they act,  $\mathbb{E}^{d|\delta}$  will denote a group of super translations with underlying super manifold  $\mathbb{R}^{d|\delta}$ .

**A.3. Geometric families.** Compare (a) in [BF86] and (1.1) in [Fre87]

**Definition A.3.** A *geometric family of manifolds over  $M$*  is

- (1) a smooth fibration of oriented manifolds  $\pi: M \rightarrow B$
- (2) a Riemannian metric on the fibers, i.e., a metric on  $T(M/B)$
- (3) a projection  $P: TX \rightarrow T(M/B)$ .

A geometric family of manifolds over  $M$  has a connection on  $T(M/B)$  gotten by fixing an arbitrary metric  $g_Y$  on the base  $M$ . Then the metric on the fibers  $T(M/B)$  along with the horizontal lift of  $g_Y$  using  $P$  gives a metric on  $X$  that has a Levi-Civita connection. Then, [Bis86] Theorem 1.9 shows that the projection of this Levi-Civita connection to  $T(M/B)$  is independent of  $g_Y$ .

**A.4. Modular forms and Eisenstein series.** An *oriented 2-dimensional lattice* is a homomorphism  $\Lambda: \mathbb{Z}^2 \rightarrow \mathbb{R}^2 \cong \mathbb{C}$  such that the ratio of the image of the generators  $\ell$  and  $\ell'$  defines a point  $\frac{\ell}{\ell'} \in \mathfrak{h} \subset \mathbb{C}$  in the upper half plane. Let  $L$  denote the smooth manifold of these lattices; we have an evident diffeomorphism  $L \cong \mathbb{C}^\times \times \mathfrak{h}$  that sends a pair of generators  $\ell, \ell'$  to  $(\ell, \ell/\ell') \in \mathbb{C}^\times \times \mathfrak{h}$ . There is an action of  $\mathbb{C}^\times \times \text{SL}_2(\mathbb{Z})$  on  $L$  by

$$\left( \mu, \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ell, \ell' \right) \mapsto (\mu^2(al + b\ell'), \mu^2(c\ell + d\ell')), \quad \mu \in \mathbb{C}^\times, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}).$$

**Definition A.4.** *Weak modular forms of weight  $n/2$*  are holomorphic functions  $f$  on  $L$  that are  $\text{SL}_2(\mathbb{Z})$ -invariant and have the property that  $f(\mu \cdot \Lambda) = \mu^{-n} f(\Lambda)$  for  $\mu \in \mathbb{C}^\times$ . Taking products of holomorphic functions gives a graded ring, denoted  $\text{MF}$  whose degree  $n$  piece, denoted  $\text{MF}_n$ , are the weight  $n/2$  weak modular forms. Define  $\text{MF}^n := \text{MF}_{-n}$ .

For  $k > 1$ , the  $2k^{\text{th}}$  Eisenstein series is

$$E_{2k}(\ell, \ell') = \sum_{n, m \in \mathbb{Z}_*^2} \frac{1}{(n\ell + m\ell')^{2k}}$$

where  $\mathbb{Z}_*^2$  denotes pairs  $(m, n) \in \mathbb{Z}^2$ , not both zero. We define the  $2^{\text{nd}}$  Eisenstein series in terms of the conditionally convergent series,

$$E_2(\ell, \ell') = \sum_{n \neq 0} \frac{1}{(n\ell)^2} + \sum_{n \in \mathbb{Z} \setminus 0} \sum_{m \in \mathbb{Z}} \frac{1}{(m\ell + n\ell')^2}$$

We denote this sum that is first over  $m$  and then over  $n$  by

$$E_2(\ell, \ell') = \sum_{(n, m) \in \mathbb{Z}_*^2}^{\text{ren}} \frac{1}{(m\ell + n\ell')^2}.$$

**A.5.  $\zeta$ -super determinants of invertible operators over geometric stacks.** For a Fredholm operator  $D$  with discrete spectrum  $\{\lambda_k\}_{k \in \mathbb{Z}}$ , following Ray–Singer [RS71, RS73] we define the  $\zeta$ -function,

$$\zeta_D(s) = \sum \lambda_k^s.$$

In a wide range of examples, this defines a holomorphic function in  $s$  for  $\text{Re}(s) \ll -1$  that can be analytically continued to a meromorphic function on  $\mathbb{C}$  that is regular at  $s = 0$ . We define the  $\zeta$ -determinant as

$$\det_\zeta(D) := \exp(\zeta'_D(0)).$$

The  $\zeta$ -Pfaffian is a square root of the  $\zeta$ -determinant, which we take to be  $\exp(\frac{1}{2}\zeta'_D(0))$  in this paper. For operators that act on  $\mathbb{Z}/2$ -graded vector bundles, we define the  $\zeta$ -super determinant as

$$\mathrm{sdet}_\zeta(D) := \frac{\mathrm{pf}_\zeta(D|_{\mathrm{odd}})}{\det_\zeta(D|_{\mathrm{even}})^{1/2}}.$$

The  $\zeta$ -super determinant can also be applied to a family of operators parametrized by a smooth manifold  $M$ , where each  $\lambda_k \in C^\infty(M)$  and  $\det_\zeta(D) \in C^\infty(M)$ . This procedure has an evident generalization to  $M$  a super manifold by Taylor expanding  $\lambda_k^{-s}$  in odd variables, then computing the  $s$ -derivative at zero termwise in the Taylor series.

Suppose we have a family of operators  $D$  over a geometric (super) stack  $\mathcal{X}$  with atlas  $u: U \rightarrow \mathcal{X}$ , i.e., for each  $S$ -point of  $\mathcal{X}$ , we have a family of operators over  $S$ , and for isomorphisms of  $S$ -points we have isomorphisms between the bundles on which these operators act that are suitably compatible with the operators themselves. Suppose further that this is a family of *invertible* operators, meaning they are invertible at each  $S$ -point. Then on  $U$  the  $\zeta$ -super determinant  $\mathrm{sdet}_\zeta(p^*D)$  defines a non-vanishing function. Isomorphisms of  $\mathcal{X}$  transform this nonvanishing function into a different nonvanishing function, which defines a cocycle for a line bundle on  $\mathcal{X}$  with respect to the atlas  $U$ . Furthermore, (by definition)  $\mathrm{sdet}_\zeta(D)$  is a section of this line bundle. This the *determinant line bundle* of  $D$  over  $\mathcal{X}$ .

*Remark A.5.* Determinant lines in the sense above are compatible with those defined by Bismut–Freed [BF86] and Quillen [Qui85], and in fact the trivialization of the determinant line when pulled back to  $U$  can be regarded as a *metric* trivialization.

Below we will encounter some  $\zeta$ -super determinants that are only *conditionally* convergent. To fix the value of the determinant, we choose a grading on the space on which the operator acts, which specifies a choice of ordering for the summation defining  $\zeta_D(s)$ , and hence for  $\zeta'_D(0)$ .

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